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ELEMENTARY GEOMETRICAL OPTICS

BY

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LONDON
G. BELL AND SONS, LTD.
1914

Cambridge:
PRINTED BY JOHN CLAY, M.A.
AT THE UNIVERSITY PRESS

PREFACE

FOR some time past I have felt the need of a text-book on Geometrical Optics with a suitable collection of examples for students preparing for Part I of the Mathematical Tripos. This book has been prepared to meet this need.

It is not desirable that a text-book on the theory of a subject should attempt to convey to students such practical knowledge as they ought to gain by experience in a laboratory, and this book therefore contains only so much account of experiments as may perhaps stimulate students to try for themselves.

The formulae for successive refractions and for series of lenses have been expressed in terms of distances rather than inclinations, such being simpler for beginners. Experience has shewn me that there is less mental confusion and liability to error when only one direction is regarded as the positive direction, rather than when the positive direction changes sense at each passage through a lens; I have therefore adopted the former method.

The book contains a somewhat wider range than the present schedule for Part I of the Tripos.

It is hardly to be hoped that a book written and seen through the press during considerable pressure of other work will prove to be free from errors, and I shall be grateful to anyone who will point them out.

A. S. RAMSEY.

11 *July*, 1914.

CONTENTS

CHAPTER I.

INTRODUCTION.

ARTICLES		PAGES
1	Light	1
2	Rectilinear propagation	1, 2
3	The pinhole camera	2
4	Velocity of light	3
5	Fizeau's method	3, 4
6	Foucault's method	4
7	Change of medium	5

CHAPTER II.

REFLECTION OF LIGHT.

8	Law of reflection	6
9	Experimental verification	6
10—12	Incident and reflected rays	7, 8
13, 14	Images	8, 9
15	Field of view in a plane mirror	9
16	Parallel mirrors	10
17	Inclined mirrors	11, 12
18	The kaleidoscope	12
19	Deviation produced by two plane mirrors	13
20, 21	Three-dimensional problems solved by geometry	13, 14
	Examples	14--17

CHAPTER III.

REFRACTION.

22	Laws of refraction	18
23	Refractive index	18, 19
24, 25	Second law	19
26, 27	Experimental verifications	20
28	Ray passing through parallel plates	20, 21
29	Total reflection. Critical angle	21

ARTICLES		PAGES
30	Total reflection prism	22
31	Luminous jet of water	22
32	Formation of images	22, 23
33	Experiment for refractive index	23
34	Focal lines	24
35	Refraction through a plane plate	25
36, 37	Deviation increases with angle of incidence	25—27
38	Examples	27, 28
39—41	Prisms	28, 29
42	Minimum deviation	29
43	Geometrical method	30
44	Field of view through a prism	30
45	Images	30, 31
46	Examples	31, 32
47	Experiments for angle of prism and its refractive index	33
48	Three-dimensional problems	33, 34
49	Geometrical method	34
	Examples	35—37

CHAPTER IV.

REFLECTION AND REFRACTION AT SPHERICAL SURFACES.

50	Definitions and conventions	38
51	Reflection at a spherical mirror	39, 40
52	Geometrical focus	40
53, 54	Images and magnification	41
55, 56	Principal focus and focal plane	42, 43
57, 58	Construction of image of a small object	43—45
59	Examples	45, 46
60	Refraction at a spherical interface	46, 47
61	Principal foci and focal planes	47, 48
62, 63	Images and magnification	48, 49
64, 65	Convergence and divergence	49
66	Construction of image of a small object	50
67	Helmholtz's formula for linear magnification	51
	Examples	51—54

CHAPTER V.

LENSES.

68	Definition	55
69	Formulae for a thin lens	55—57
70, 71	Divergent and convergent lenses	58, 59
72	Opticians' measure of power	59

ARTICLES		PAGES
73	Relation between conjugate points and foci	59
74—76	Construction of image of a small object	60—62
77	Angular magnification	62
78	Longitudinal magnification	62
	Examples	62—65

CHAPTER VI.

COMBINATIONS OF TWO OR MORE THIN LENSES.

79	Two thin lenses	66, 67
80	Principal foci and focal planes	67
81, 82	Linear magnification. Unit points and planes	68, 69
83	Focal length of combination	69
84	Distances from unit points	70
85	Divergent and convergent combinations	70, 71
86—88	Geometrical constructions	71—73
89	Equivalent lens	73, 74
90, 91	Numerical examples	74, 75
92, 93	Any number of thin lenses on the same axis	75—77
	Examples	77—80

CHAPTER VII.

THICK LENSES.

REFRACTION THROUGH A NUMBER OF MEDIA SEPARATED BY COAXIAL SPHERICAL SURFACES.

94—96	Direct incidence on a solid sphere, and Examples	81—84
97—101	A thick lens	84—87
102	Different forms of lenses	87—89
103, 104	Nodal points	89—91
105	Optical centre of a lens	91, 92
106	Linear magnification	92
107, 108	Examples	93—95
109—113	Experiments for cardinal points	95—97
114—116	Any number of media separated by coaxial spherical surfaces	97, 98
117	Combination of two instruments	98, 99
	Examples	99—102

CHAPTER VIII.

DISPERSION AND ACHROMATISM.

ARTICLES		PAGES
118—122	Newton's experiments	103—105
123—125	Pure spectrum. Dark lines	105, 106
126	Dispersive power	106, 107
127	Irrationality of dispersion	107
128—131	Achromatism. Dispersion produced by prisms	107—109
132—134	Achromatism of lenses	109, 110
135—137	Object-glasses and eye-pieces	111, 112
138	General system of lenses	112, 113
	Examples	113—116

CHAPTER IX.

ILLUMINATION.

139, 140	Intensity of emission	117, 118
141	Intrinsic brightness	118
142—145	Illumination	118—120
146	Objects appear equally bright at all distances	121
147	Brightness of an image formed by a refracting system	121, 122
148	Apparent brightness	122
	Examples	123—125

CHAPTER X.

MISCELLANEOUS THEOREMS.

149	Stationary reduced path	126, 127
150, 151	Theorem of Malus. Orthotomic surfaces	127, 128
152	Focal length in terms of aperture and thickness	128, 129
153—157	Caustics and aberration	129—131
158	Visible images	131
159	Thin pencils. Focal lines	132
160	Formulae for primary and secondary foci	133, 134
161	Circle of least confusion	134
	Miscellaneous examples	134—136

CHAPTER XI.

THE EYE AND VISION.

162	The eye	137, 138
163	Formation of images on the retina	138
164	Defects of vision	139—141
165, 166	Vision through a lens. Simple microscope	141, 142

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ARTICLES		PAGES
167, 168	Telescopes and telescopic systems	143, 144
169, 170	Apparent distance	144, 145
171—173	Magnifying power	145—147
174	Eye-ring and entrance pupil	147, 148
175, 176	Magnifying power of a telescope in normal adjustment	148
	Examples	149, 150

CHAPTER XII.

OPTICAL INSTRUMENTS.

177	The common astronomical telescope	151
178—180	Magnifying power and field of view	152—154
181—183	Radius of stop. Eye-ring. Cross wires . . .	154, 155
184	Adjustment of the eye-piece	155, 156
185	Examples	156, 157
186	Defects of object-glasses and eye-pieces . . .	157
187	Eye-pieces	157, 158
188	Ramsden's eye-piece	158, 159
189	Huyghens' eye-piece	159—161
190	Examples	161
191	Galileo's telescope	161, 162
192	Field of view	162—164
193	Adjustment for vision at a finite distance . .	164
194, 195	Opera glasses. The prism binocular	165
196	Examples	165, 166
197—201	Reflecting telescopes	166—168
202, 203	The compound microscope	168, 169
204	Forms of object-glasses	169, 170
205	Telephotographic objective	170
	Examples	171—173

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ELEMENTARY GEOMETRICAL OPTICS

CHAPTER I

INTRODUCTION

1. THE existence of distant objects is revealed by the sense of sight. Whether an object is visible or not does not depend only on the eyes of the observer but also, in everyday language, on whether there is sufficient *light* by which to see it. Light is therefore an agent by which vision is conditioned. The question "what is light?" has been the subject of speculative theory. Newton and others believed and maintained that so-called *luminous* bodies emit in all directions a continuous stream of infinitesimal particles which are propagated with enormous velocity and impinging on the eye produce the sensation known as vision. The undulatory theory of light, in which the phenomena are explained as being due to waves in a medium originated with Huyghens*. The medium was supposed to be an elastic solid filling all space, so porous in character as to offer no resistance to the passage of material bodies through it and yet so rigid as to transmit vibrations with great rapidity. The elastic solid theory has now given place to the theory that light consists of electromagnetic waves, but the nature of the displacements that constitute the waves leaves room for further investigation.

Many facts concerning light can be investigated without any hypothesis about its nature by the aid of a few laws based on experiment and by simple geometrical considerations. This part of the science is called *Geometrical* Optics and to this we shall confine our attention in the chapters that follow.

2. Rectilinear propagation of light.

If a pinhole is made in a piece of cardboard and placed between the eye and a candle flame, the flame is only visible through the pinhole, when the latter lies on a straight line between the eye

* *Traité de la lumière*, Leyden, 1690.

and the flame. This and similar experiments, together with the formation of clear-cut shadows of opaque bodies, lead to the hypothesis that light is propagated in straight lines. To such an extent is the rectilinear propagation of light accepted as a matter of experience that the ordinary method of testing the straightness of a line is to look along it. We shall assume therefore that *in a homogeneous medium light travels in straight lines.* Closer investigation shews that there are cases in which this statement is not strictly true and in certain circumstances light bends round obstacles in its path, but we shall not be concerned with such cases.

Any straight line along which light travels is called a *ray.* When light from a distant source passes through a small aperture, such as, for example, sunlight entering a darkened room through a small hole in a shutter, the beam of light that comes through the aperture is called a *pencil.* The form of the pencil depends on the form of the aperture, and every such pencil has a finite cross section. When we speak of a *ray of light* we mean a pencil whose cross section is so small that we may represent the pencil by a straight line. Newton* defined a ray as "The least Light or part of Light, which may be stopt alone without the rest of the Light, or propagated alone, or do or suffer anything alone, which the rest of the Light doth not, or suffers not."

3. The Pinhole Camera.

The rectilinear propagation of light is well illustrated by the apparatus known as the pinhole camera.

It can be constructed by making a hole O about 1 mm. in diameter in the centre of the bottom of a box. If the lid is replaced by a sheet of ground glass AB, inverted images are formed on the glass of objects the light from which passes through the hole, in the manner shewn in the diagram.

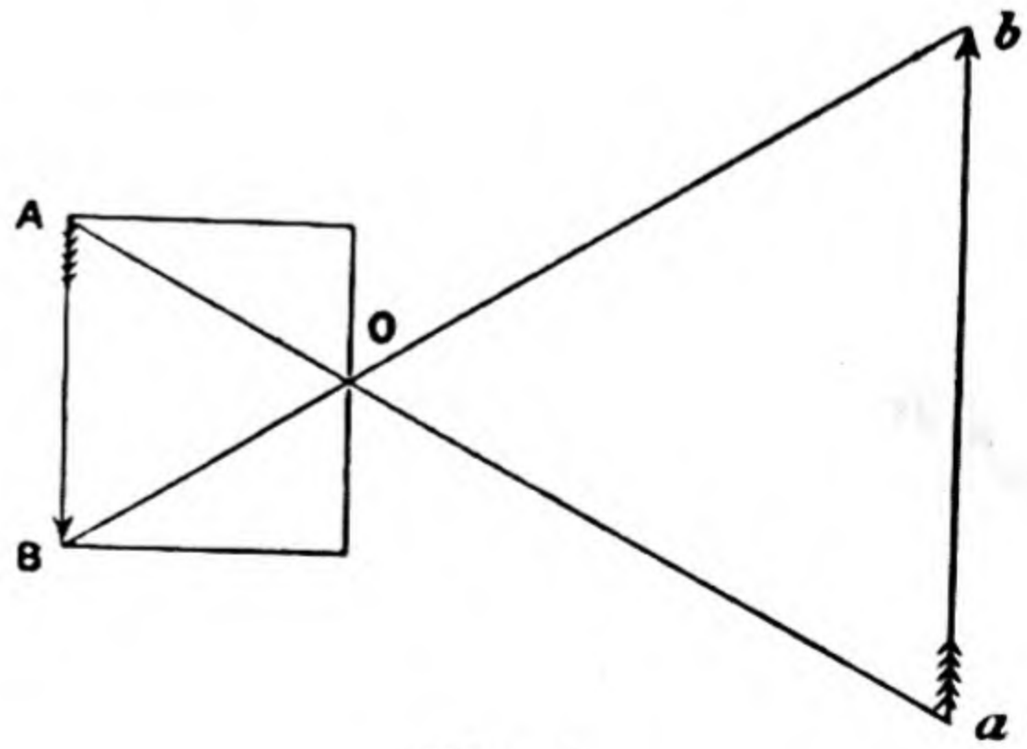


Fig. 1.

* "Opticks or a treatise of the reflexions, refractions, inflexions and colours of light," London, 1704.

4. The Velocity of Light.

That light has a definite velocity was first demonstrated by Olaf Römer, a Danish astronomer (1675). He found that the apparent time of revolution of one of Jupiter's satellites varies with the position of the earth in its orbit, decreasing or increasing according as the earth is moving towards or away from Jupiter. Jupiter has a satellite whose period is a little more than two days so that frequent observations of its period can be made. These were made when the earth and Jupiter were in opposition, and again when in conjunction, and it was found that an eclipse of the satellite occurred 996 seconds sooner than the predicted time. This difference was ascribed to the velocity of light, 996 seconds representing the time that it would take to travel a distance equal to the diameter of the earth's orbit. This determination gives $2{\cdot}967 \times 10^{10}$ cms. per second as the velocity of light.

5. Fizeau's Method.

The magnitude of the velocity of light renders laboratory experiments on it difficult to devise, but a determination was made by Fizeau in 1845* using terrestrial distances only.

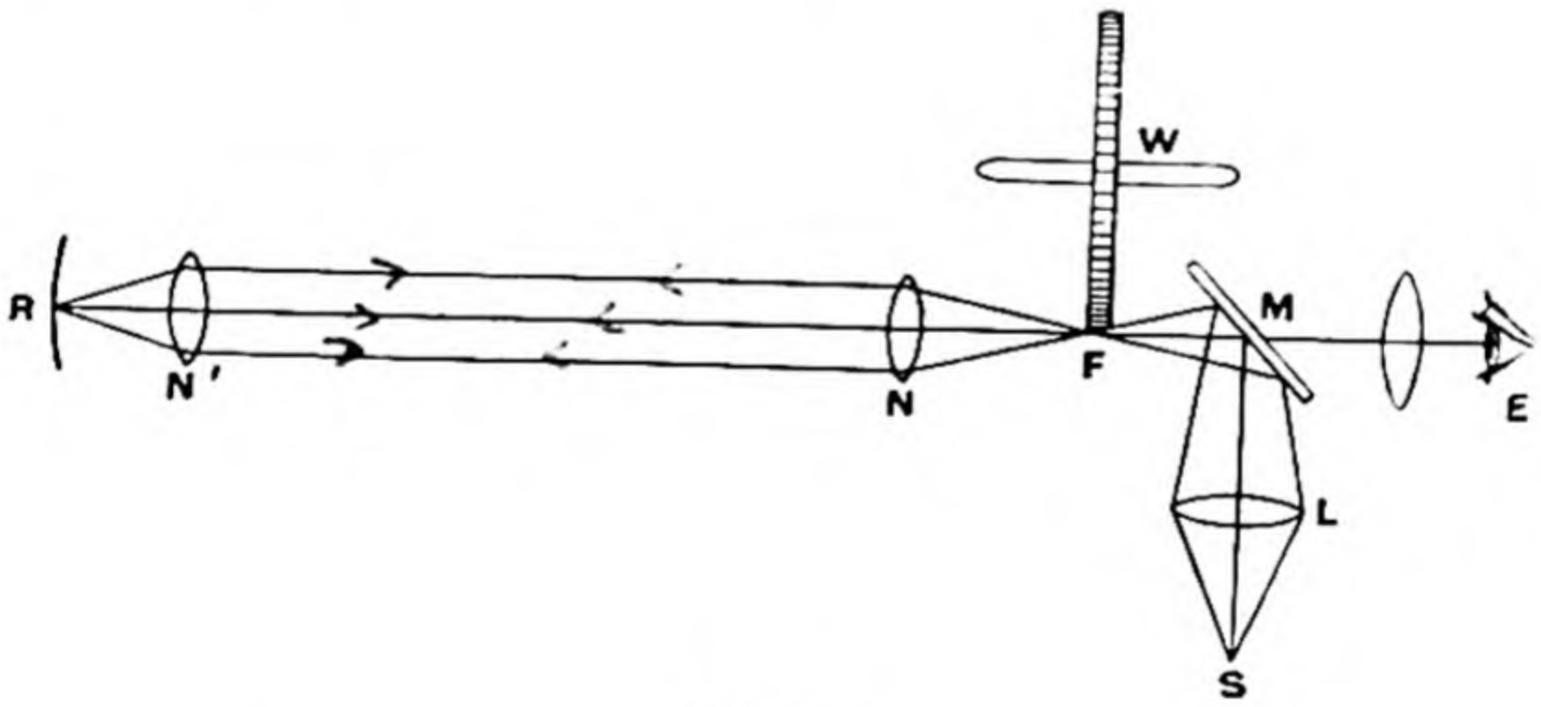


Fig. 2.

A convergent lens L and a plane glass plate M form an image at F of a bright source of light S (Fig. 2), F is also the focus of the lens N so that after passing N the rays are parallel. They pass through a similar lens N' at a considerable distance and are reflected back again over the same path by a mirror R. In Fizeau's actual experiment the distance NN' was 8·633 kms. At F is the rim of a toothed wheel W so arranged that a tooth obstructs the light and a gap allows it to pass. When the wheel

* H. Fizeau, *Comptes Rendus de l'Académie des Sciences*, t. xxix. p. 90, 1849.

is rotated slowly the light alternately appears and disappears, and as the speed increases the light becomes visible continuously owing to persistence of vision. As the speed is increased still further a point is reached when the light disappears altogether, this is when the speed is such that the time taken for the light to travel from F to R and back is also the time in which a gap in the wheel is supplanted by a tooth. If the speed be doubled the light reappears, if trebled it disappears again and so on. Fizeau used a wheel with 720 teeth and the first disappearance occurred when the speed was 12·6 rotations per second, making the velocity of light $3{\cdot}13 \times 10^{10}$ cms. per second.

With better apparatus in 1874 Cornu* obtained the result $2{\cdot}9995 \times 10^{10}$ cms. per second.

6. Foucault's Method.

Another method was devised by Foucault† in 1850 and subsequently modified and used by Michelson in 1880, Newcomb in 1882 and others.

Light from a slit S falls on a rapidly revolving mirror O. During each revolution the reflected light falls on a distant mirror M and is reflected back along the same path to O. If in the time the light takes to travel from O to M and back the mirror O turns through an angle α, the reflected ray at O will make an angle 2α with OS. This reflected beam of light passes through a lens L and forms an image of the slit S at S'. By measuring SS' and OS the angle 2α can be determined and then if the rate of rotation of the mirror O is known we get the time taken for light to travel from O to M and back and thus find the velocity of light.

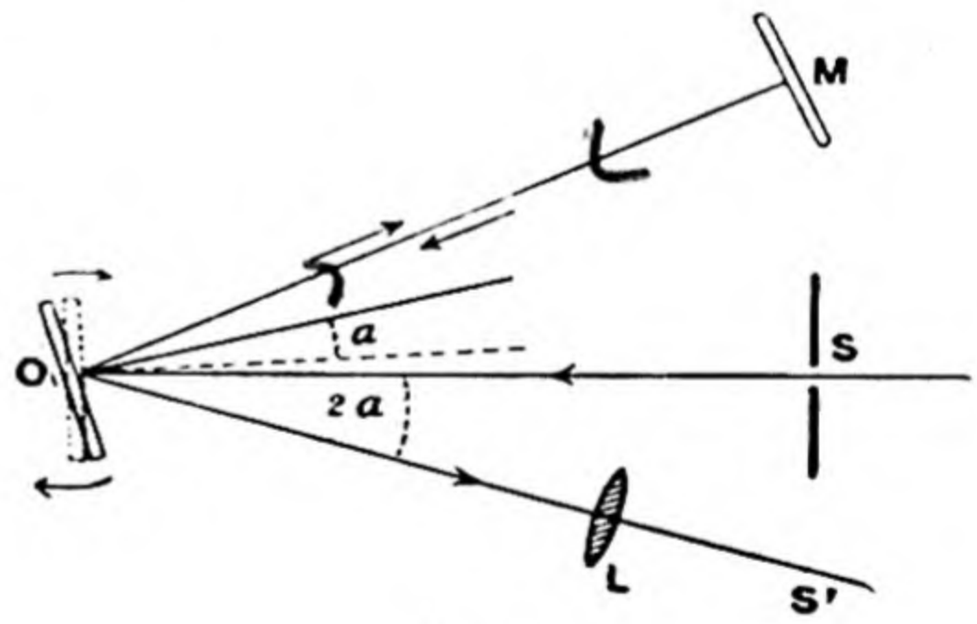


Fig. 3.

These investigations will be found more fully described in books on physical optics to which this subject really belongs. A combination of the methods of Fizeau and Foucault in which the

* A. Cornu, *Comptes Rendus de l'Académie des Sciences*, t. LXXIX. p. 1361, 1874.
† L. Foucault, Poggendorff's *Annalen der Physik u. Chemie*, b. 81, p. 434, 1850.

toothed wheel was replaced by a grating and a revolving mirror was applied by Michelson* in 1902.

7. When a pencil of light falls on the surface of a body or on the surface of separation of two different media the light may be reflected from the surface back into the first medium, or it may pass through the surface in the same or in an altered direction, in the latter case it is said to undergo *refraction*; or it may be partly reflected and partly refracted and there may also be some scattering of light at the surface, as when light falls on a ground glass surface.

Reflection and Refraction take place in accordance with certain laws based on experiment and it will be our object to investigate the consequences of these laws in simple cases.

* A. A. Michelson, "The velocity of light," *Phil. Mag.* (6) III. p. 330, 1902.

CHAPTER II

REFLECTION OF LIGHT

8. Law of Reflection.

When a ray of light is reflected the incident and reflected rays are in the same plane with the normal to the reflecting surface at the point of incidence and make equal angles with the normal.

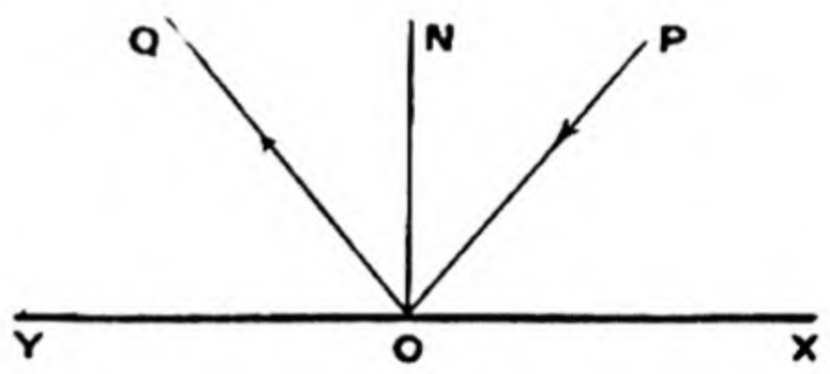


Fig. 4.

Thus if ON is the normal at the point of incidence O of the ray PO, the reflected ray OQ lies in the plane PON and the angles PON, NOQ are equal.

9. This law permits of simple experimental verification as follows :

Lay a sheet of paper on a horizontal table and place upon it a plane mirror with its plane XY vertical. Stick two pins AP, BQ upright in the table and so that $AP = BQ$. Place the eye in such a position that the pins as seen by

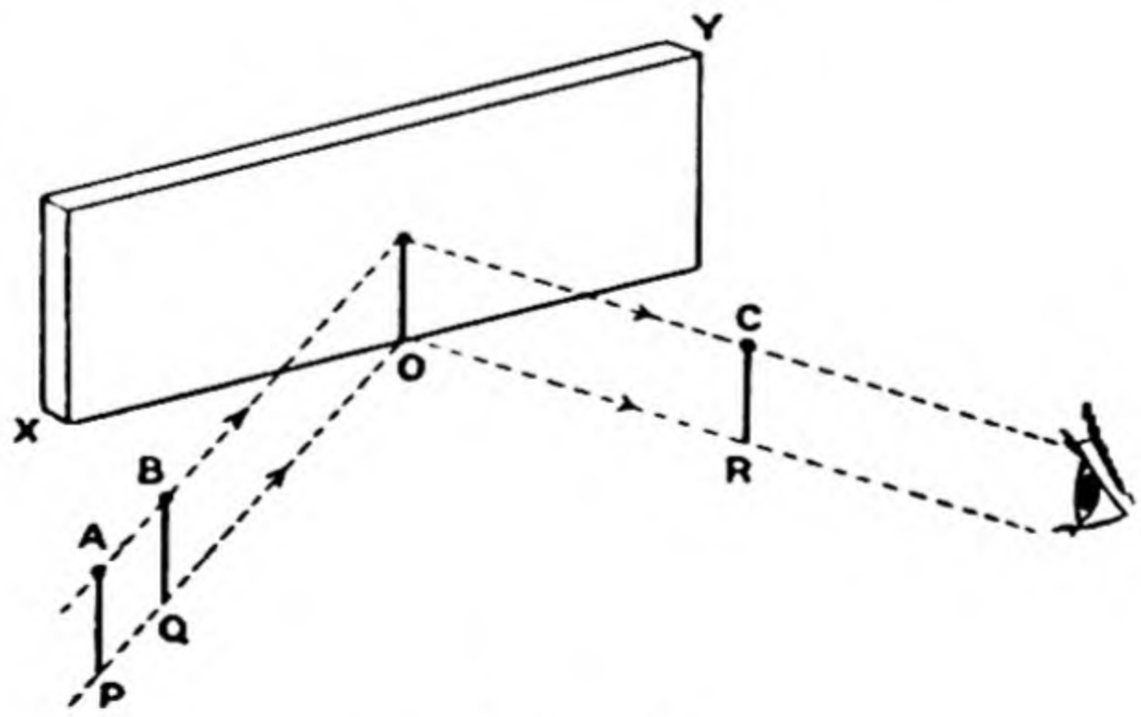


Fig. 5.

reflection appear to coincide. Stick up another pin CR on the line of vision, and adjust its height so that C appears to coincide with A and B. It will be found that $CR = AP = BQ$, and this verifies the first part of the law. If the line PQ be ruled meeting the mirror in O and OR be joined, the second part of the law follows by measuring the inclination of PO, OR to the mirror.

10. The projections of an incident and reflected ray on any plane through the normal at the point of incidence are equally inclined to the normal.

In the figure PO, OQ are the incident and reflected rays, and $P'O$, OQ' their projections on any plane through the normal ON. If through any point N on the normal a plane $PP'NQ'Q$ is drawn at right angles to the normal cutting the rays and their projections in P, Q, P', Q', the line $P'NQ'$ is the projection of PNQ on the plane $P'OQ'$.

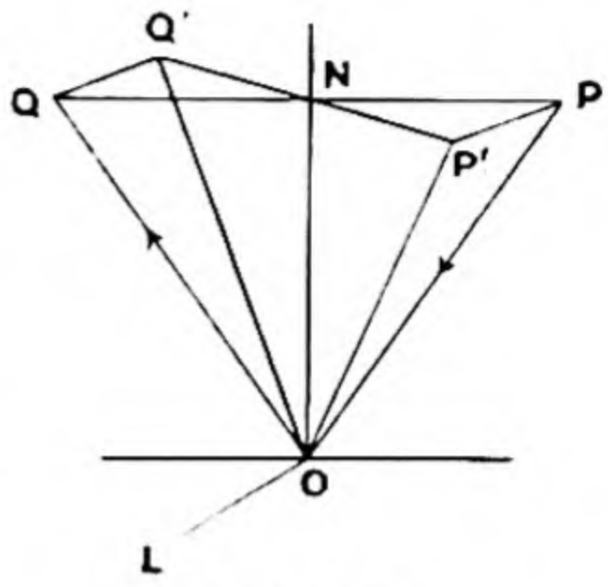


Fig. 6.

Hence P' is the projection of P and Q' of Q, and the angles $PP'N$, $PP'O$, $QQ'N$, $QQ'O$ are all right angles.

In the right-angled triangles ONP, ONQ

the angle $$PON = NOQ,$$

therefore $$OP = OQ \text{ and } PN = NQ.$$

In the right-angled triangles $PP'N$, $QQ'N$

the angle $$PNP' = QNQ', \text{ and } PN = NQ;$$

therefore $$PP' = QQ' \text{ and } NP' = NQ'.$$

In the right-angled triangles ONP', ONQ'

$$NP' = NQ',$$

therefore the angle $P'ON = NOQ'$, which was to be proved.

11. The incident and reflected rays are equally inclined to any straight line in the tangent plane at the point of incidence.

With the figure and construction of the last article we have in the right-angled triangles $PP'O$, $QQ'O$

$$OP = OQ \text{ and } PP' = QQ',$$

therefore the angle $$POP' = QOQ';$$

that is, the incident and reflected rays are equally inclined to any plane through the normal. Hence if OL is at right angles to the plane $P'OQ'$, the incident and reflected rays are equally inclined to OL and therefore to any line in the tangent plane to the reflecting surface at O.

12. The incident and reflected rays are equidistant from any point in the tangent plane at the point of incidence.

If the reflected ray OQ be produced to P' in the direction QO, it is clear that the lines OP, OP' are symmetrically placed with regard to the mirror, or with regard to the tangent plane to it at O, if it be not a plane mirror, and therefore the distances of the lines OP, OP' from any point in this plane are the same.

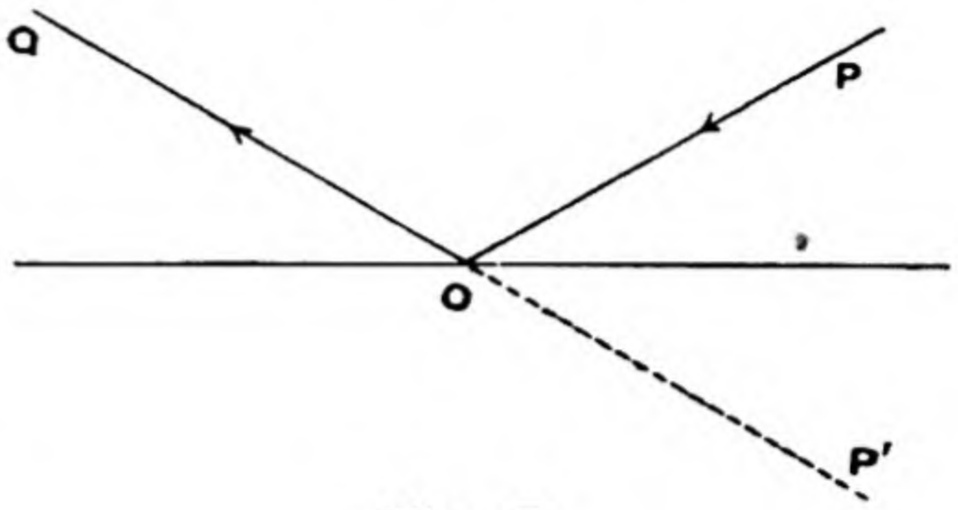


Fig. 7.

13. Reflection at a plane surface. Formation of Images.

A consequence of the law of reflection is that all rays emanating from a point A and falling on a plane mirror XY will proceed after reflection as though they came from a point A' behind the mirror and equidistant from it. Hence if PQ be the pupil of an eye conveniently placed to receive some of the rays, it will receive a cone of rays of vertex A' and base PQ; that is, the eye will see the point A'. The point A' is called the *image* of A in the mirror XY.

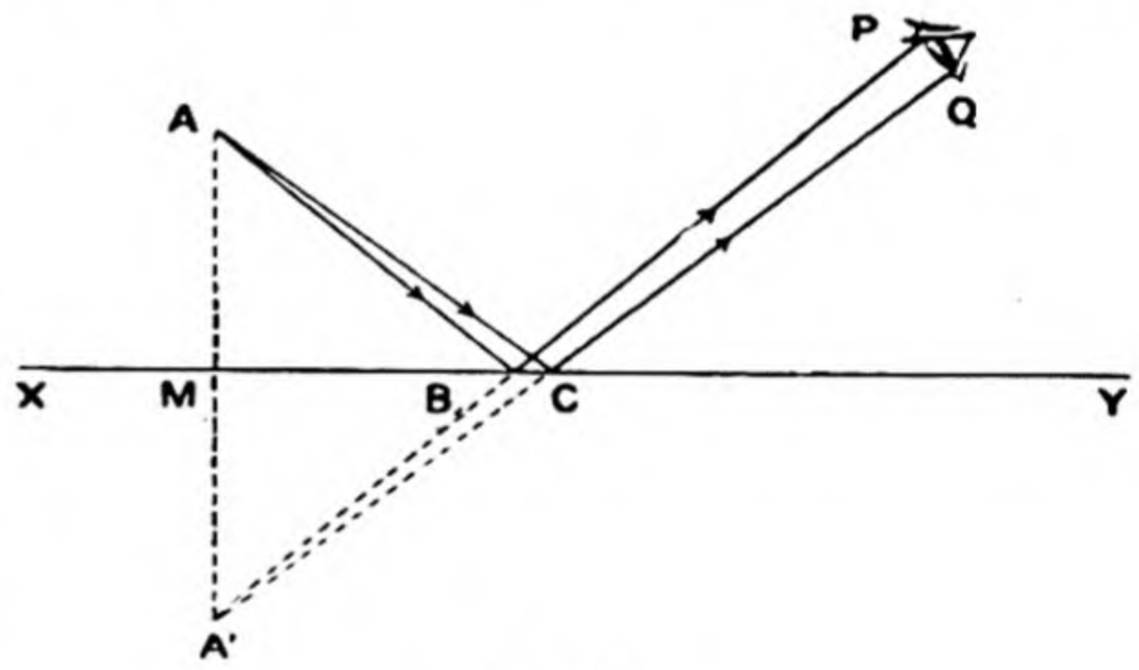


Fig. 8.

The image is in this case called a *virtual image* because the rays do not pass through it, but only their prolongations.

To construct the actual paths of the rays, the image A' is first found by making $MA' = MA$, then the cone $A'PQ$ is broken by the mirror in BC and bent back to A.

14. Image of an object.

In like manner the image $A'B'$ of an object AB in the mirror XY can be constructed. The image is virtual, equal to the object and the two are symmetrically placed with regard to the mirror.

Regarding the mirror as a vertical plane, if the object is erect so is the image, but it is *perverted*, that is, if to an observer

looking between the object and the mirror A is on the left hand and B on the right, to the same observer A' is on the right hand

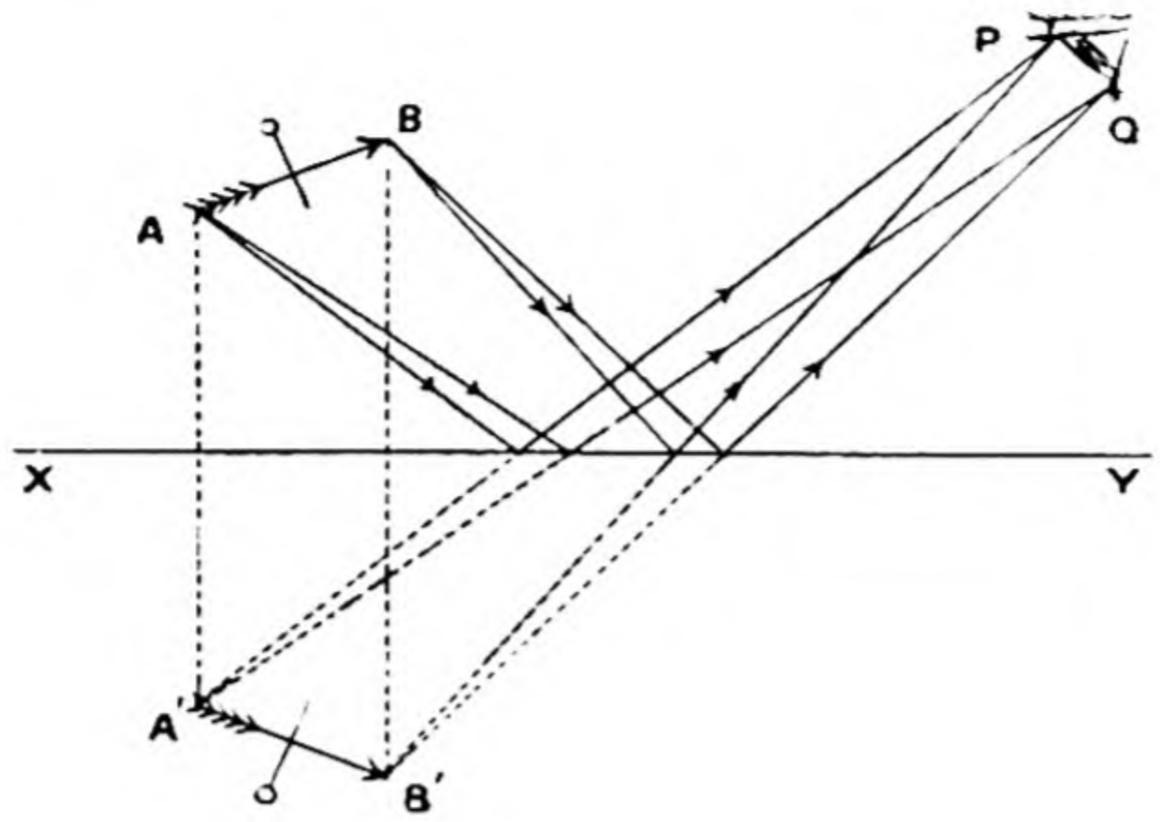


Fig. 9.

and B' on the left. This perversion is easily illustrated by viewing a page of print as reflected by a mirror.

15. Field of view in a plane mirror.

If the direction of a ray be reversed it will retrace its path. Just as rays diverging from a point E appear after reflection to proceed from an image E', so all rays converging after reflection to

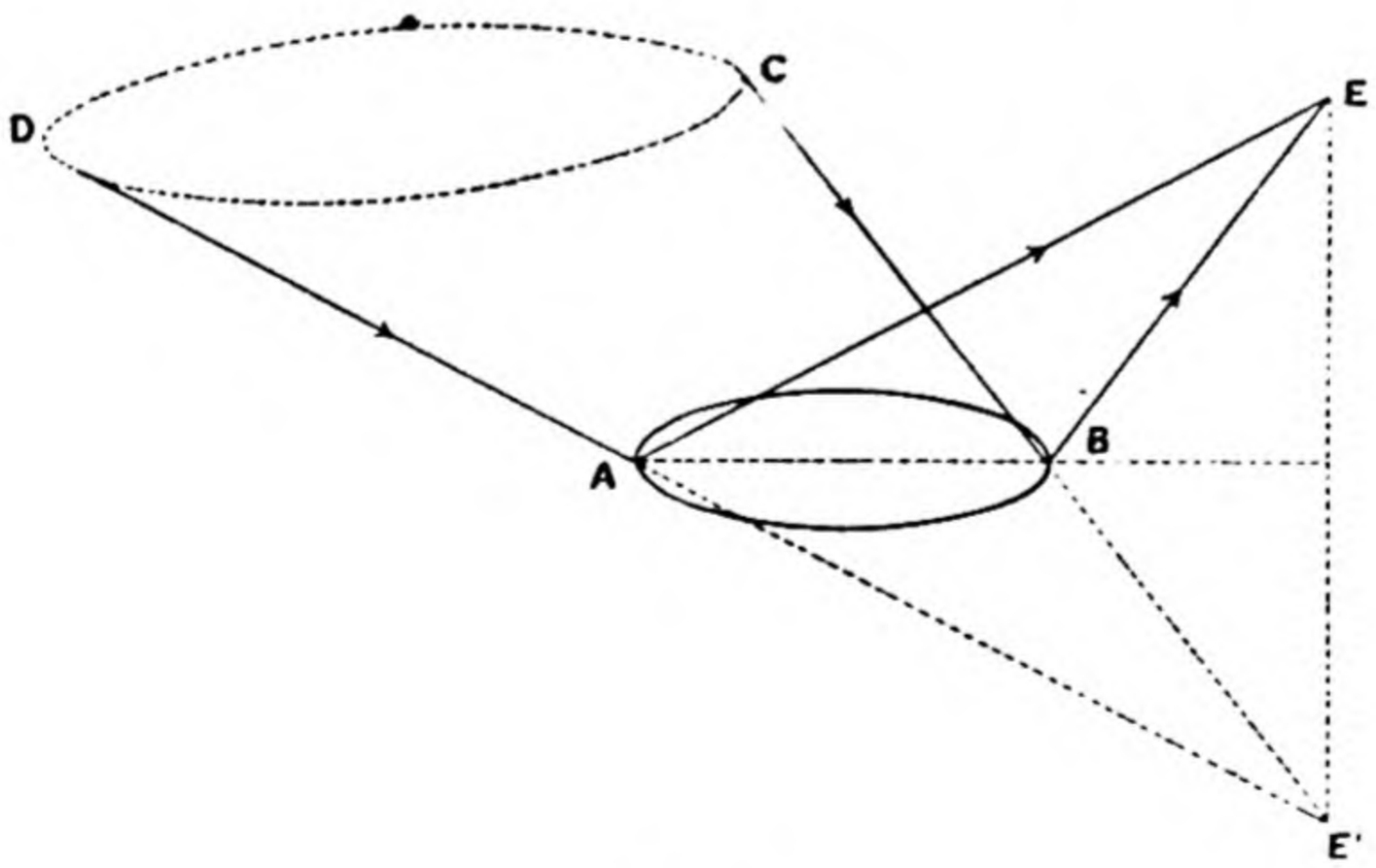


Fig. 10.

an eye at E must be directed before reflection to the image E' of E. Hence the field of view visible at E in the mirror AB is found by producing the cone whose vertex is E' and base AB as shewn in the figure.

16. Parallel Mirrors.

If O be an object between two parallel mirrors AA, BB, there are two infinite series of images. Thus

B_1 is the image of O in BB
B_2 „ „ B_1 in AA
B_3 „ „ B_2 in BB

and so on.

Also
A_1 is the image of O in AA
A_2 „ „ A_1 in BB
A_3 „ „ A_2 in AA

and so on.

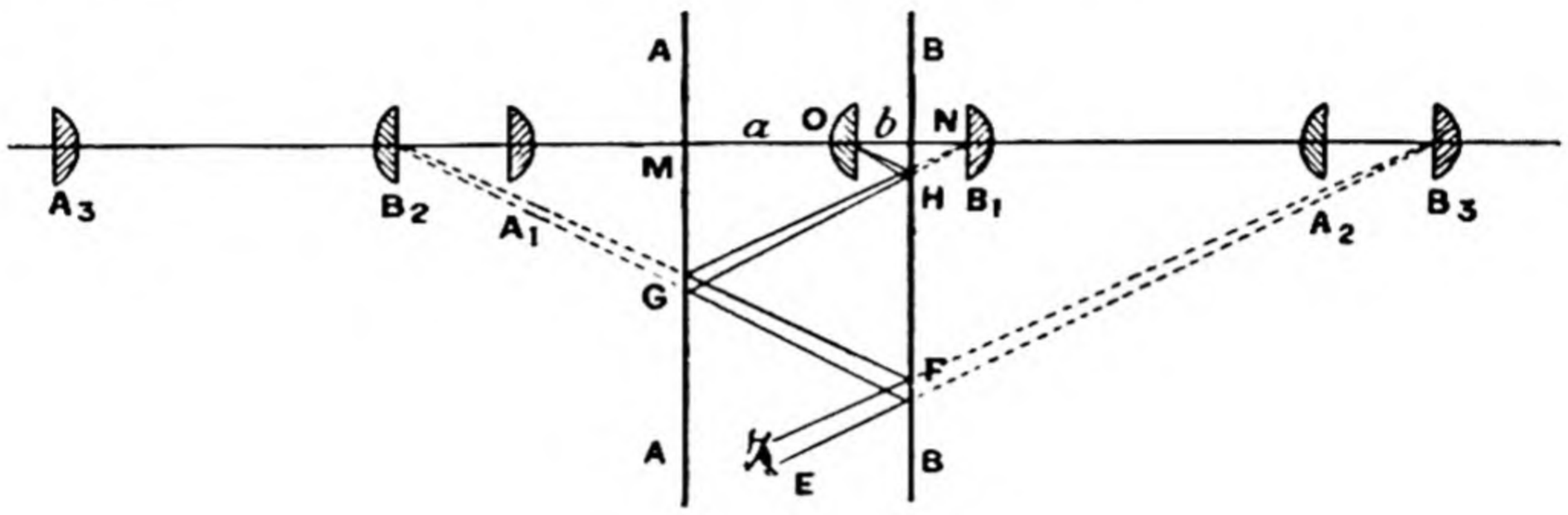


Fig. 11.

Again if a, b denote the distances of O from the mirrors we have

$$NB_1 = NO = b, \qquad MA_1 = MO = a,$$
$$MB_2 = MB_1 = a + 2b, \qquad NA_2 = NA_1 = 2a + b,$$
$$NB_3 = NB_2 = 2a + 3b, \qquad MA_3 = MA_2 = 3a + 2b,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots$$

whence it follows that

$$B_1B_3 = B_2B_4 = B_3B_5 = \ldots = 2(a + b)$$

and

$$A_1A_3 = A_2A_4 = A_3A_5 = \ldots = 2(a + b).$$

That is, the distance between consecutive images of either series behind either mirror is twice the distance between the mirrors.

To construct the path of a ray by which an eye between the mirrors sees the object after, say, two reflections in the mirror BB and one in the mirror AA. Join a point of the pupil E to B_3 cutting BB in F, join FB_2 cutting AA in G, join GB_1 cutting BB in H, and join HO. Then $OHGFE$ is the path of the ray required.

17. Inclined Mirrors.

Let P be a bright point between two plane mirrors, the figure representing a section by a plane through P perpendicular to the line of intersection of the mirrors meeting them in OA, OB. Since the line joining a point and its image is bisected by the mirror that forms the image it is clear that all the images of P lie on a circle whose centre is O.

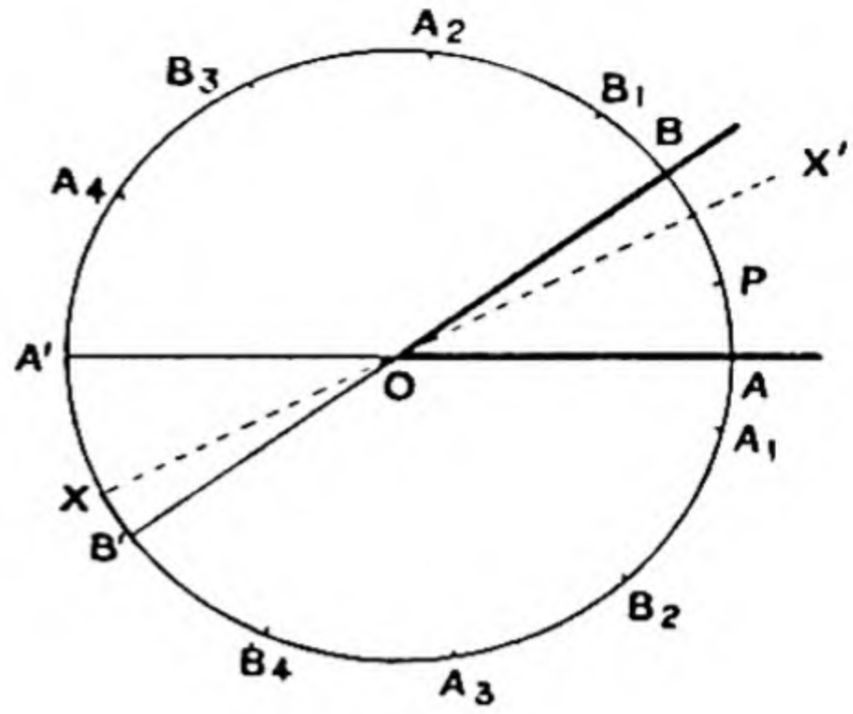


Fig. 12.

Then we have series of images as follows:

A_1 is the image of P in OA, B_1 is the image of P in OB

A_2 „ „ A_1 in OB, B_2 „ „ B_1 in OA

A_3 „ „ A_2 in OA, B_3 „ „ B_2 in OB

....., ...

Again if α, β be the arcual distances of P from A and B, and $\alpha + \beta = i$, we have

$$PA_1 = 2\alpha$$
$$PA_2 = \beta + BA_2 = \beta + BA_1 = 2\beta + 2\alpha = 2i$$
$$PA_3 = \alpha + AA_3 = \alpha + AA_2 = 2\alpha + 2i$$
$$PA_4 = \beta + BA_4 = \beta + BA_3 = 2\beta + 2\alpha + 2i = 4i$$
$$PA_5 = \alpha + AA_5 = \alpha + AA_4 = 2\alpha + 4i$$

and so on.

That is, there is a set of images A_1, A_3, A_5, ... at angular distances 2α, $2\alpha + 2i$, $2\alpha + 4i$, ... from P in the clockwise sense; and a set of images A_2, A_4, A_6, ... at angular distances $2i$, $4i$, $6i$, ... from P in the counterclockwise sense. These sets continue until an image falls on the arc $A'B'$ behind both mirrors and then stop since rays proceeding from such an image clearly cannot undergo further reflection at either mirror. The image A_{2n+1} falls on the arc $A'B'$ if

$$PAA_{2n+1} > PAB', \text{ or if } 2\alpha + 2ni > \pi - \beta,$$

i.e. if $$(2n+1)\,i > \pi - \alpha \text{ or } 2n + 1 > (\pi - \alpha)/i.$$

The image A_{2n} falls on the arc $A'B'$ if

$$PBA_{2n} > PBA', \text{ or if } 2ni > \pi - \alpha,$$

i.e. if $$2n > (\pi - \alpha)/i.$$

Hence the number of images in this series is the integer next greater than $(\pi - \alpha)/i$.

In the same way there is another series $B_1, B_2, B_3, \ldots$ of images formed by taking first the image in OB and the number of these in like manner is the integer next greater than $(\pi - \beta)/i$.

If $i = \pi/m$ where m is an integer, the integers next greater than $(\pi - \alpha)/i$ and $(\pi - \beta)/i$ are both m. In this case the points $PA_2A_4A_6 \ldots B_6B_4B_2$ are corners of a regular polygon of m sides, and the points $A_1A_3A_5 \ldots B_3B_1$ are corners of an equal polygon, and one corner of the first or the second polygon lies on the arc $A'B'$ according as m is even or odd. Hence the two final images of each series coincide and the total number including the bright point P is $2m$.

To an eye situated anywhere between the two mirrors the images that fall behind one mirror only will all be visible. But an image that falls behind both mirrors is not visible for some positions of the eye. If X be such an image and XO be produced to X', then if X be the result of a final reflection in OA it will not be visible to an eye within the angle BOX', because the line joining X to the eye would not intersect the mirror OA. Similarly if X be the result of a final reflection in OB, it will not be visible to an eye within the angle AOX'.

18. The Kaleidoscope.

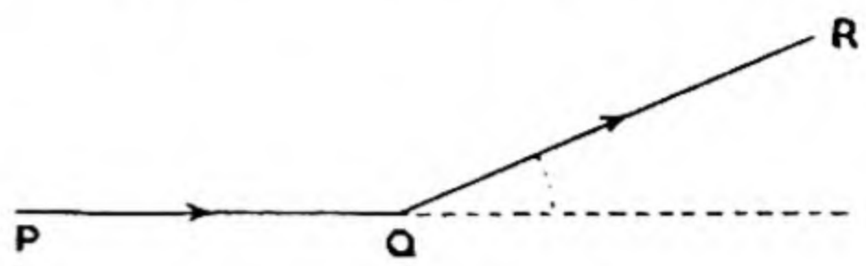
If two plane mirrors intersecting at an angle of 60° are fixed into a cylindrical tube with its axis parallel to the line of intersection of the mirrors, and an eye looking down into the cylinder views bright objects placed within it such as small pieces of coloured glass, the eye will also see five images symmetrically ranged round the line of intersection of the mirrors. By shaking the objects can be rearranged and many patterns of hexagonal form can be obtained.

More elaborate series of images can be produced by using three plane mirrors placed in the form of a prism whose cross section is an equilateral triangle.

The mirrors might be placed at other angles than 60°, but in order to get a symmetrically placed set of images the angle must be a submultiple of 180°.

19. Deviation produced by two plane mirrors.

By *deviation* is meant the angle through which a ray is bent; thus if PQR is a ray bent at Q the deviation is the angle between QR and the production of PQ, *not* the angle PQR.



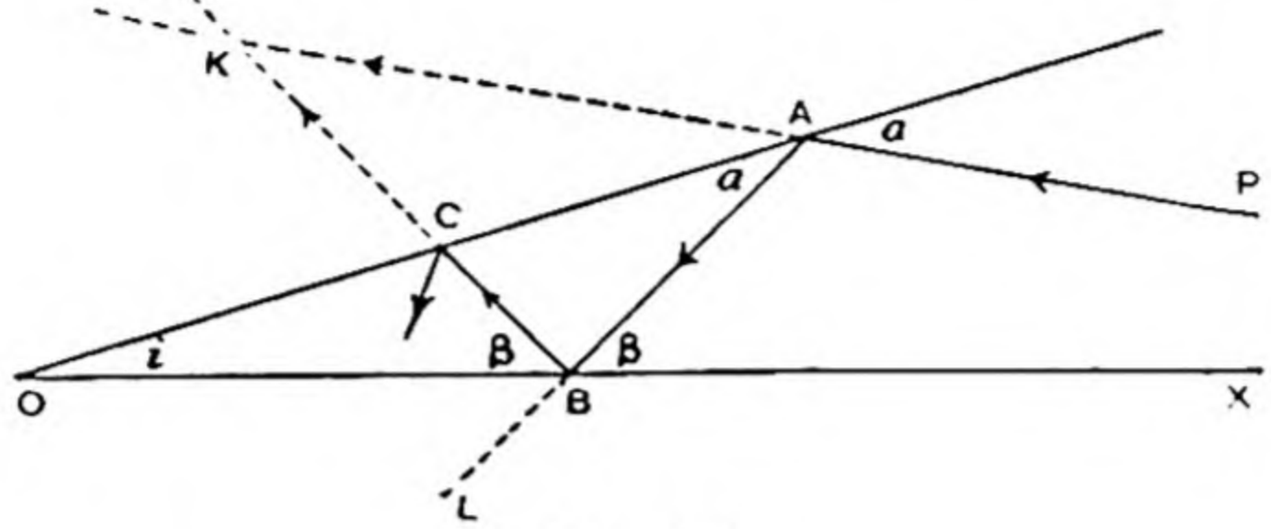


Fig. 13.

If $PABC\ldots$ be the path of a ray in a plane at right angles to the line of intersection of two plane mirrors inclined at an angle i; and PA meet BC in K, the deviation produced by two reflections is the angle

$$\begin{aligned} AKB &= KBL - KAB \\ &= 2\beta - 2\alpha \\ &= 2\,(ABX - BAO) = 2i. \end{aligned}$$

That is, each pair of reflections produce a deviation equal to twice the angle between the mirrors.

20. Three-dimensional problems.

When the path of a ray does not lie wholly in one plane, the principles that determine the direction of the ray after a number of reflections are those of Arts. 10, 11, *i.e. that the projections of an incident and reflected ray on any plane through the normal at the point of incidence obey the law of reflection*, and that *an incident and reflected ray are equally inclined to any straight line in the tangent plane at the point of incidence.*

A method which yields simple solutions of some problems in three dimensions is based on the use of a sphere as follows:

If OP be a radius of a sphere drawn parallel to an incident ray and from P a chord PQ of the sphere is drawn parallel to the normal at the point of incidence, then since the angles OPQ, OQP are equal the radius OQ must be parallel to the reflected ray.

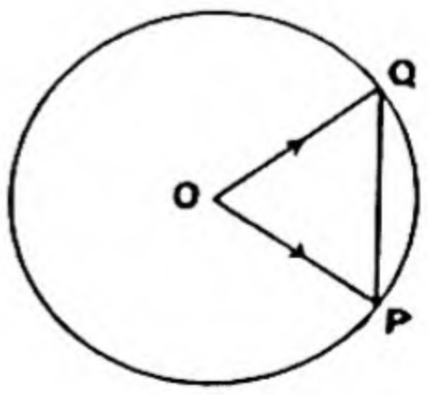
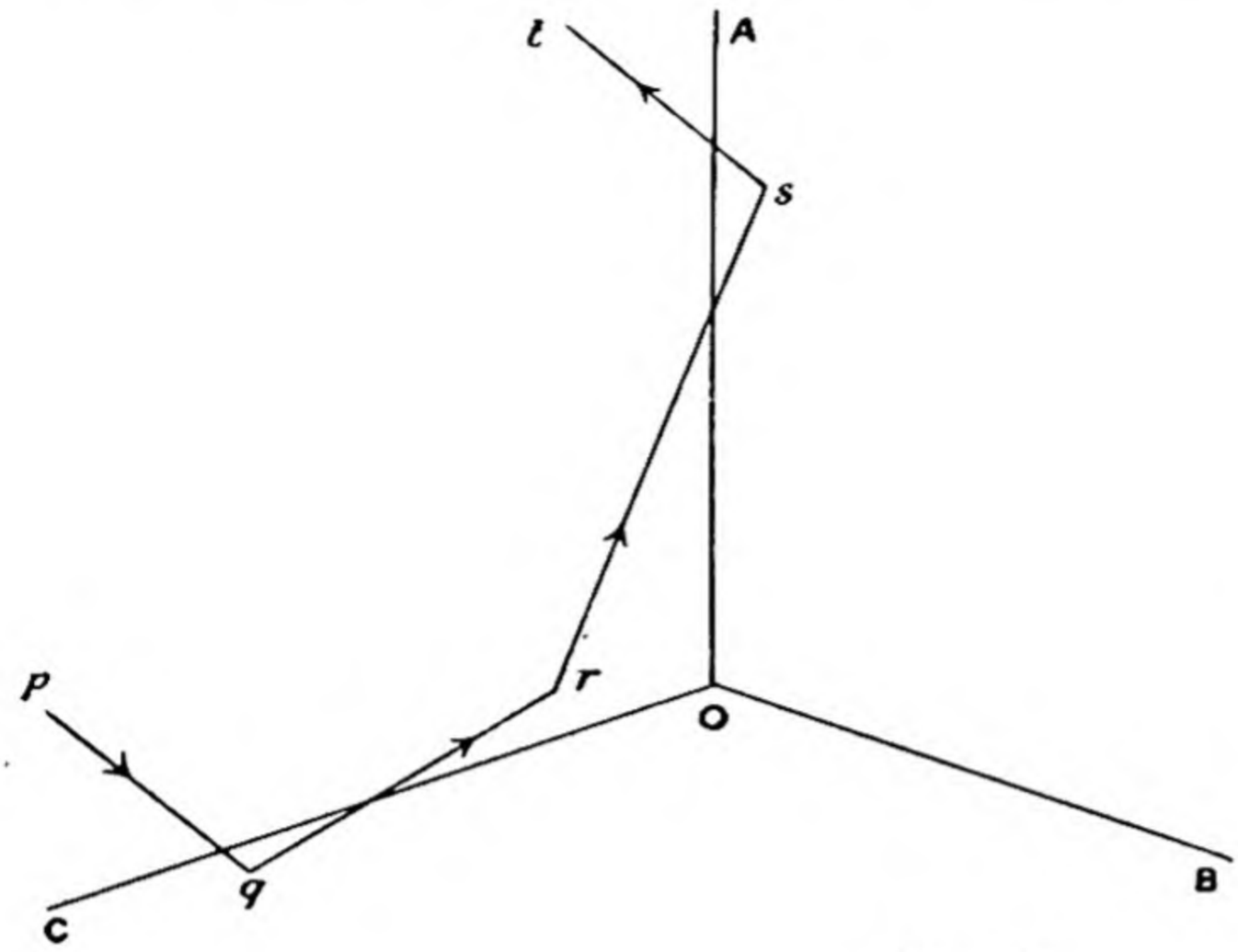


Fig. 14.

21. Example. *When a ray is reflected in succession at three plane mirrors mutually at right angles its final direction is parallel to its initial direction.*



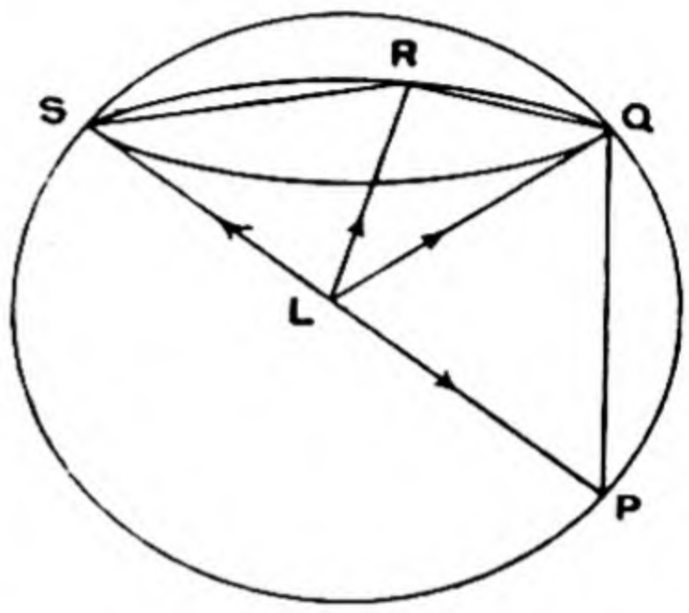


Fig. 15.

The ray *pqrst* strikes the mirrors *BOC*, *COA*, *AOB* in succession at *q*, *r*, *s*. From the centre *L* of a sphere a radius *LP* is drawn parallel to *pq*; a chord *PQ* is drawn parallel to the normal *OA* to the mirror *BOC*, and therefore *LQ* is parallel to *qr*. The normals *OB*, *OC* are both at right angles to *OA*, so that if chords *QR*, *RS* of the sphere are drawn parallel to *OB*, *OC* they must lie in the plane section of the sphere at right angles to *PQ*, and since *BOC* is a right angle so is *QRS*; therefore *QS* is a diameter of the circle *QRS* and *PS* is a diameter of the sphere. But *LR* is parallel to *rs* and *LS* to *st*, and *PLS* being a straight line *st* must be parallel to *pq*.

EXAMPLES.

1. If a ray falls on a plane mirror which turns about a straight line at right angles to the ray, shew that the angle turned through by the reflected ray is twice the angle of rotation of the mirror.

2. If a ray is reflected at the three sides of a triangle *ABC* so that its path is a triangle *PQR*, shew that *PQR* must be the pedal triangle of the triangle *ABC*.

3. If a luminous point be placed at the focus of a reflecting ellipse, shew that after two reflections any ray will return to the focus from which it started.

4. Find the angle between two plane mirrors if a ray parallel to the first becomes after two reflections parallel to the second.

5. Explain how, with a point source of light, a plane mirror and a vertical rod, a horizontal shadow can be thrown on a vertical screen. (M. T. i. 1908.)

6. A bright point moves in a straight line between two plane mirrors; shew that each image also describes a straight line. (Coll. Exam. 1909.)

7. Shew by a diagram how a man by means of two mirrors can when looking straight in front of him see the back of his head. Trace the course of a pencil of rays from the back of his head to his eye. (Coll. Exam. 1891.)

8. If a man standing at the centre of a square room faces a corner formed by two reflecting walls and shuts one eye, will he be able to see the reflection of the whole of his face? (Coll. Exam. 1912.)

9. The inner surfaces of the sides BC, CD, DA of a square $ABCD$ can reflect light, and E, F are two small holes in AB. Find the position of a point P outside the square such that a ray PE after reflection at the three sides may emerge through F and again pass through P. (M. T. 1906.)

10. Two persons stand at places A and B, and C is the middle point of the lower edge of a vertical mirror of length $2c$ which rests on the ground; prove that each person can see the other in the mirror, provided

$$ab \sin \psi < c\,(a \sin \theta + b \sin \phi),$$

where a, b are the lengths of AC, BC, θ, ϕ are the angles they make with the mirror, and ψ is either the sum or the difference of these angles according to circumstances. (The mirror is supposed to be sufficiently high.)
(M. T. 1902.)

11. Shew that by means of two plane mirrors a ray of light may be made to pass through any given point in any given direction in an infinite number of ways. (Coll. Exam. 1889.)

12. A ray of light passes in a plane normal to two inclined plane mirrors. Prove that the straight lines of which its path consists all touch the same circle.

13. An object lies between two mirrors placed at right angles. An eye observes the object directly, and on turning through a right angle observes it again doubly reflected. Shew that the eye must be at some point on a fixed sphere. (Coll. Exam. 1906.)

14. OA, OB are the sections of two plane mirrors inclined at 45° by a plane perpendicular to both and a luminous point P is placed between them so that $POA = 10°$, find the whole number of images formed.

Draw the pencil by which an eye placed within the angle AOB sees the third of one of the two sets of images. (Coll. Exam. 1896.)

15. Shew that, if the angle between two plane mirrors be 40°, there are always 9 images formed of a bright point between the mirrors; but that, if the angular distances of the eye from the mirrors both exceed the angular distance of the bright point from the bisector of the angle, only 8 can be seen. (Trinity Coll. 1890.)

16. Two plane mirrors are inclined at an angle of 72°, and a candle is placed between them at an angular distance of 12° from one. Shew that an eye placed at random between the mirrors is equally likely to be able to see three or four or five images of the candle. (M. T. 1909.)

17. If a man standing in a given position between two mirrors inclined to each other at an angle a can see himself full face after three reflections, find the direction which he must face. (Coll. Exam. 1906.)

18. A rod of length $2a$ inclined at an angle θ to the horizon is viewed by an eye, in a vertical plane through the rod, directly and by reflection in still water, so that the rod and its image appear to be of the same length. If h and k are the heights of the centre of the rod and the eye above water-level, shew that the horizontal distance of the eye from the centre of the rod is

$$h \tan\theta \pm \sqrt{h^2 \sec^2\theta - k^2 + a^2},$$

where both values are admissible if the image can be seen.

(Coll. Exam. 1908.)

19. Two parallel mirrors of height h are at a distance b apart, and a luminous point is placed at the foot of one. A screen is placed perpendicularly to the mirrors at a height k above their highest points; shew that if $k > h$, there will be on the screen alternate bright and dark bars of breadths $b(1+k/h)$ and $b(k/h-1)$ respectively. Shew that if $k < h$, but $> \frac{1}{3}h$, there will be brighter and darker bars of breadths $b(1-k/h)$ and $b(3k/h-1)$ respectively. (Trinity Coll. 1891.)

20. A man in a rectangular room whose walls are plane mirrors sees his image after reflection at the four walls in succession. Shew that if he walks towards his image it will go directly away from him and will move at the same rate that he does. Which side of himself will he see?

(Coll. Exam. 1911.)

21. If a number of mirrors form the faces of a pyramid, shew that if a ray is reflected successively any number of times by the mirrors, all the reflected rays touch a single sphere. (Coll. Exam. 1902.)

22. If a ray of light is reflected once at each of the six faces of a rectangular parallelopiped, prove that its final and initial directions are the same no matter in what order it is incident on the faces. (M. T. 1903.)

23. One end of a rectangular box is open, a ray enters and is reflected the same number of times at each side of the box and once at the closed end, prove that the emergent ray makes the same angle with the original ray as if there were but one reflection at the closed end of the box.

(Coll. Exam. 1898.)

24. Give a geometrical construction for the path of a ray in order that it may be reflected at the faces of a rectangular parallelopiped in turn at the same points and pass through a given point within the parallelopiped.

(Coll. Exam. 1905.)

25. If a ray be twice reflected in such a way that the three directions of the ray are equally inclined to one another, then the normals to the reflecting surfaces at the points of incidence must be inclined at an angle of 60°.

(Coll. Exam. 1901.)

26. A ray enters a polished circular cylinder which has one end closed by a plane reflector at right angles to its axis. Prove that if the shortest distance of the ray from the axis of the cylinder be $\cos\frac{\pi}{n}$ times the radius and the ray emerges after $2n+1$ reflections, its direction will be the same as if it was reflected once at the closed end. (Coll. Exam. 1903.)

27. Two mirrors, rigidly connected so that their planes are at right angles, are free to turn about the line of intersection of their planes. Shew that, if a ray be incident on one of the mirrors along a given straight line and successively reflected between them, then, as the mirrors are turned about their line of intersection, all the reflected rays always lie on the same surface of revolution. (Coll. Exam. 1903.)

28. A ray of light is reflected successively at two plane mirrors inclined to one another at an angle a; shew that, if a be small and if the ray be incident very nearly normally, whether it be in a plane perpendicular to both mirrors or not, the deviation is approximately $2a$. (St John's Coll. 1904.)

CHAPTER III

REFRACTION

22. Laws of Refraction.

When a ray of light passes from one medium to another the incident and refracted rays are in the same plane with the normal to the interface at the point of incidence and on opposite sides of the normal, and the sines of their inclinations to the normal bear to one another a ratio which is constant for the same two media and the same kind of light.*

If NON' is normal to the interface XY between two media and PO, OQ are the incident and refracted rays; the angles NOP, $N'OQ$ are called the angles of incidence and refraction, and denoting them by ϕ, ϕ' the law states that OP, OQ, ON are coplanar and

$$\sin \phi = \mu \sin \phi',$$

where μ is a constant depending on the nature of the media and the kind of light.

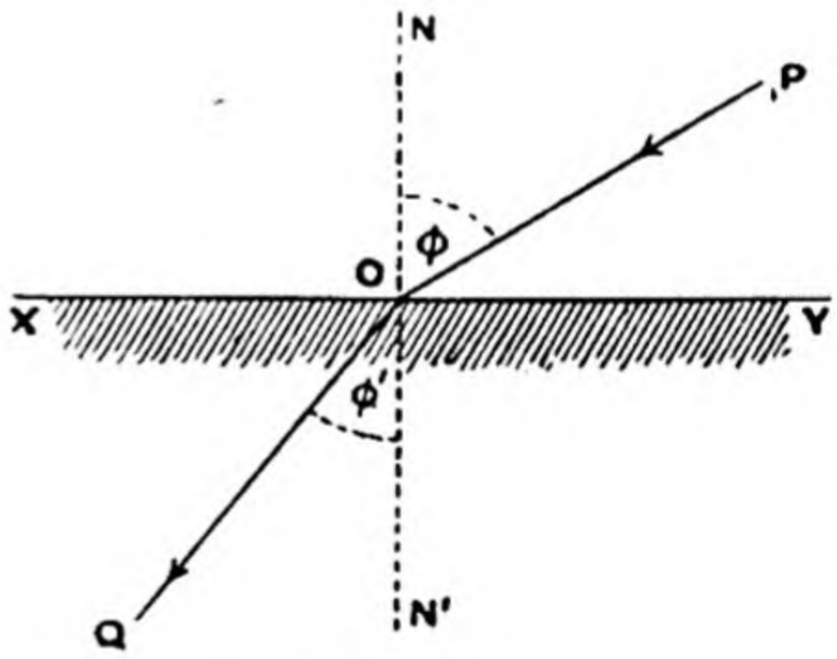


Fig. 16.

23. The constant μ is called the *refractive index* of the second medium as compared with the first.

According to the undulatory theory refractive indices of different media are *inversely proportional to the velocities of light* in the different media. If we take air as the standard medium for comparison, then the refractive index from air into any other medium, say glass, $= \dfrac{\text{velocity of light in air}}{\text{velocity of light in glass}}$.

* The law of sines appears to have been discovered by Willebrod Snell, of Leyden, who died in 1626. It was first published by Descartes in 1637, but he appears to have obtained it from Snell and the authorship of the discovery was the subject of much controversy.

Similarly the refractive index from medium A to medium B may be written ${}_A\mu_B = \dfrac{\text{velocity of light in medium } A}{\text{velocity of light in medium } B}$;

and in like manner for light passing from medium B to medium A,

$${}_B\mu_A = \frac{\text{velocity of light in medium } B}{\text{velocity of light in medium } A}.$$

24. This leads to the **second law of refraction**, viz. *the index of refraction from medium A to medium B is the reciprocal of the index of refraction from medium B to medium A*; or

$${}_A\mu_B \times {}_B\mu_A = 1;$$

which is also equivalent to the statement that *if a ray of light be reversed it will retrace its path.*

Again if v_A, v_B, v_C denote the velocities of light in media A, B, C we have

$${}_B\mu_C = \frac{v_B}{v_C} = \frac{v_A}{v_C} \div \frac{v_A}{v_B} = {}_A\mu_C \div {}_A\mu_B;$$

which may be read as follows, taking A to be air:—the index of refraction from medium B to medium C is equal to the index of refraction from air to medium C divided by the index of refraction from air to medium B.

Hence if ϕ is the angle of incidence in medium B and ϕ' is the angle of refraction in medium C,

then $$\sin\phi = {}_B\mu_C \sin\phi',$$

or $${}_A\mu_B \sin\phi = {}_A\mu_C \sin\phi'.$$

25. When reference is made to "a medium of refractive index μ" without specifying any other medium it is understood that the air is the medium of comparison, and the law of sines is written

$$\sin\phi = \mu \sin\phi'$$

for light passing from air into the medium considered.

In the same way if light is passing from a medium of refractive index μ to a medium of refractive index μ', the law of sines is written

$$\mu \sin\phi = \mu' \sin\phi'.$$

Speaking generally the denser the medium the greater the refractive index. For ordinary yellow light incident in air, the refractive index of crown glass is 1·50, and of water 1·33. We

AMAR SINGH COLL

shall see in a later chapter that the values of the refractive indices depend on the colour of the light.

26. Experimental verifications.

There are many ways of verifying the laws of refraction experimentally.

The Law of Sines.

Place a rectangular block of glass on a sheet of paper on a horizontal table. Make a vertical scratch on one face—through the point P in the figure, which represents a horizontal section. Place the eye at E so that the scratch is visible through the glass. Stick two pins Q, R into the table on the line of vision. Mark or rule on the paper the position of P, the line XY which represents the surface through which the ray emerges, and the line QR which gives the emergent ray. If RQ meets XY in O, then PO is the path of the ray in the glass and the angles of incidence and refraction can then be measured and their sines compared. Or instead of measuring the angles cut off equal lengths, say OP, OR, and then the perpendiculars PN, RM from P and R to the normal will be found to be in a constant ratio.

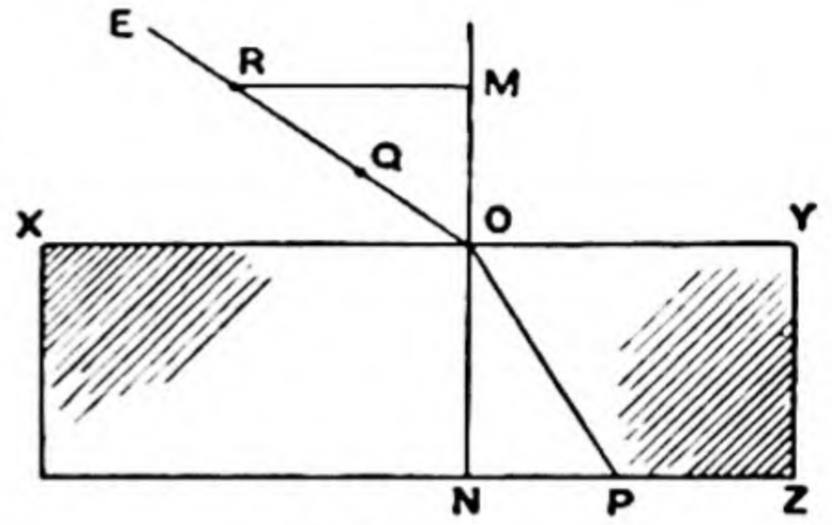


Fig. 17.

27. The Law of Reversibility.

Place on a horizontal table a block of glass with vertical parallel plane faces standing on a sheet of paper. The block is represented by XY. In a convenient position stick up two pins P, Q and place the eye E on the other side of the block so that the pins appear to coincide. Stick up two more pins R, S on the line of vision. Rule the lines PQ, RS and on removing the block they will be found to be parallel. And the faces of the block being parallel the angles made with the normal by the path of the ray in air are both ϕ and in glass both ϕ'.

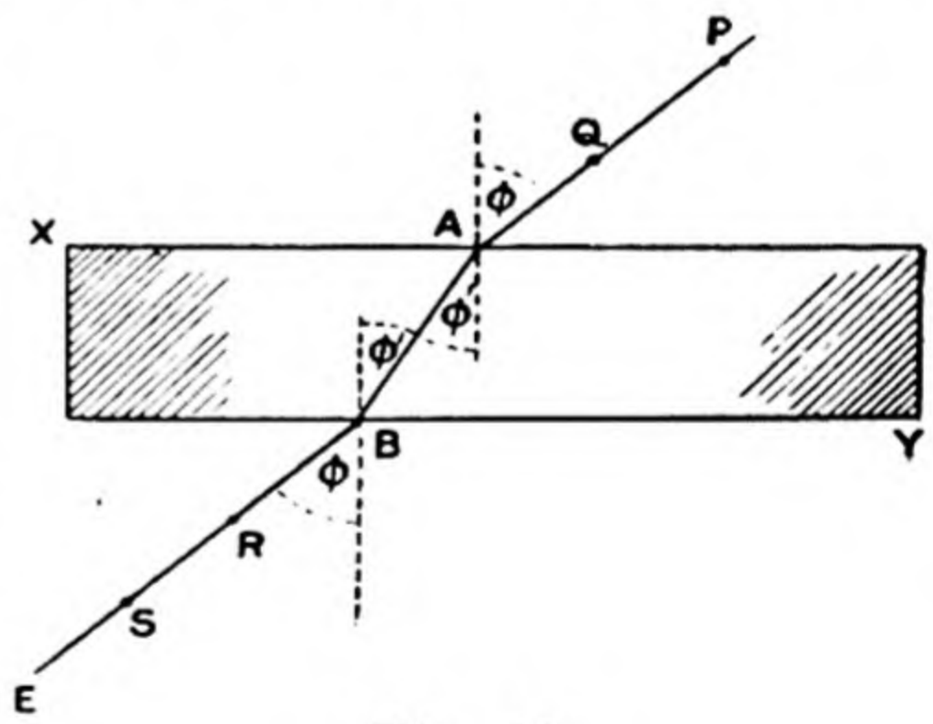


Fig. 18.

Hence the index of refraction from air to glass being $\sin\phi/\sin\phi'$ and that from glass to air being $\sin\phi'/\sin\phi$, they are reciprocals and the path is reversible.

28. *When a ray traverses a series of parallel plates and enters the original medium again the emergent ray is parallel to the incident ray.*

Let μ_a, μ_b, μ_c, μ_a be the absolute refractive indices in succession.

Then denoting angles as in the figure

$$\mu_a \sin \phi = \mu_b \sin \phi',$$

$$\mu_b \sin \phi' = \mu_c \sin \psi',$$

$$\mu_c \sin \psi' = \mu_a \sin \psi.$$

Whence we get $\phi = \psi$.

This is clearly capable of simple experimental verification.

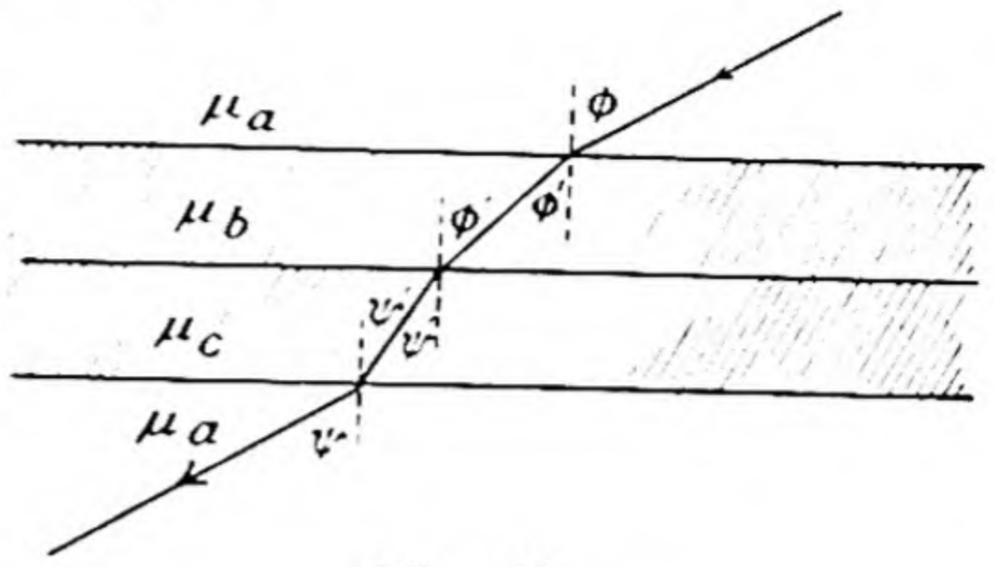


Fig. 19.

29. Total Reflection. Critical Angle.

Let μ be the refractive index of a medium optically denser than air (*i.e.* $\mu > 1$). Then if ϕ, ϕ' are inclinations of a ray to the normal in air and in the medium

$$\sin \phi = \mu \sin \phi'.$$

Now $\sin \phi$ cannot have a value greater than unity, therefore $\sin \phi'$ cannot have a value greater than $1/\mu$.

Let rays proceed in all directions from a point source of light O in the denser medium. ON is perpendicular to the interface and as the rays diverge more and more widely from ON the angle ϕ' increases and so does the corresponding ϕ, until a position is reached indicated by OP_3 in the figure for which

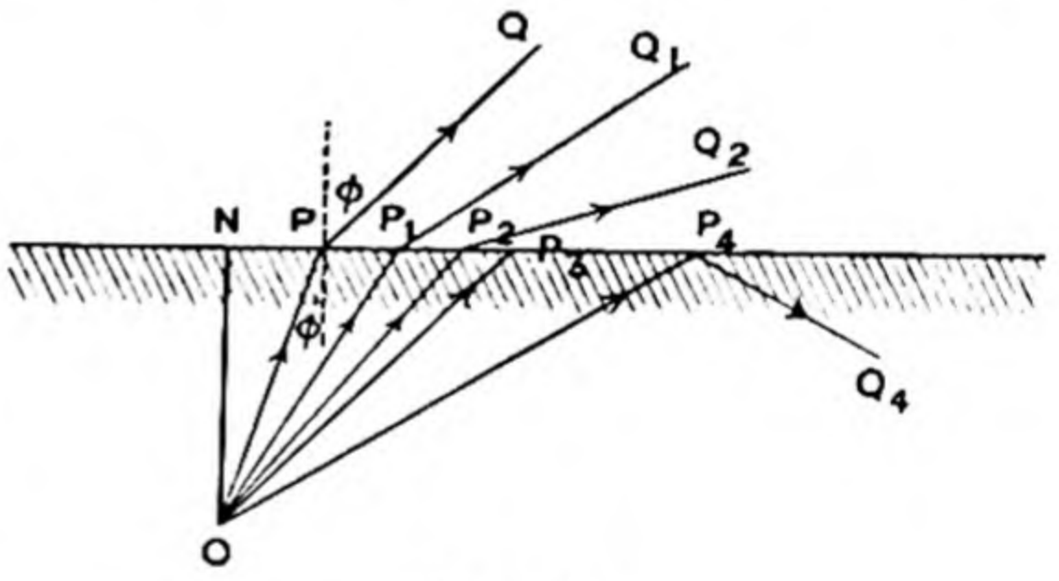


Fig. 20.

$$\phi = 90^\circ \text{ and } \phi' = \sin^{-1} \frac{1}{\mu}.$$

This ray does not emerge into air, but, after incidence on the surface of the denser medium, proceeds along the surface. Rays for which $\phi' > \sin^{-1} \frac{1}{\mu}$ are *totally reflected*, as for example OP_4Q_4 in the figure.

The angle $\sin^{-1} \frac{1}{\mu}$ is called the **critical angle*** for the given medium in relation to air; and a ray incident in a medium denser than air at the critical angle will then proceed along the surface, while a ray incident at an angle greater than the critical angle will undergo total reflection.

* Joh. Kepler, *Dioptrice*, Vienna, 1611, trans. Leipzig, 1904.

30. Total Reflection Prism.

Since the refractive index of glass is about 1·5, the critical angle is $\sin^{-1}\frac{2}{3}$ or 41° 47′. If a glass prism be constructed whose section is an isosceles right-angled triangle, rays incident normally on one of the two perpendicular faces AB pass into the prism without deviation and are incident on the slant face AC at an angle of 45° greater than the critical angle, consequently they are totally reflected at the face AC and turned through a right angle*. The rays then strike the face BC normally and emerge without further deviation.

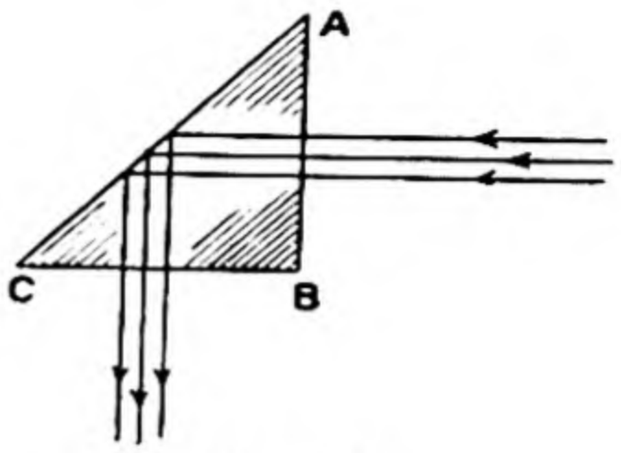


Fig. 21.

The use of such a prism for turning a beam of light through a right angle is preferable to a plane glass mirror silvered at the back, because the front surface of the glass reflects as well as the silvered surface and there may be confusion of images arising therefrom. The best mirrors are silvered on the front surface, but they are very liable to tarnish.

31. Luminous jet of water.

An experiment† which exhibits the phenomenon of total reflection effectively consists in passing a narrow beam of light into a vessel of water along the line of an issuing jet. In Fig. 22 the jet issues through a small aperture at F and a horizontal beam of light enters through glass at E. The rays that enter the jet of water are always incident on the surface of the jet at angles greater than the critical angle and therefore never emerge, so that the jet is rendered luminous through the whole of its length. The jet can be coloured by using coloured glass at E.

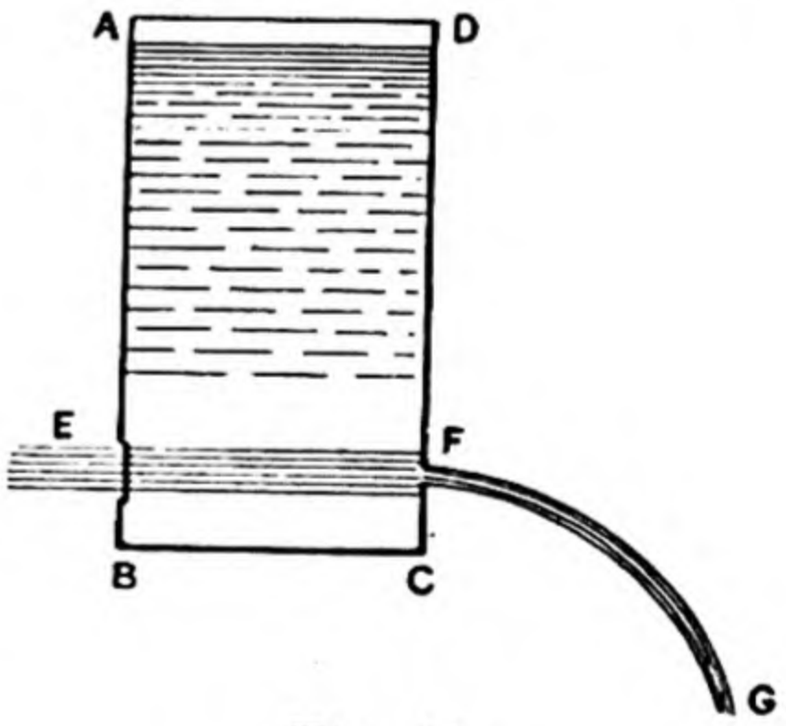


Fig. 22.

32. Formation of Images by refraction at a plane surface.

Let rays diverge from a point P and pass into a medium of refractive index μ with a plane face AR.

Let PRS be any ray and let the refracted ray RS be produced backwards to meet AP at right angles to the plane in Q. Then

* Kepler, *oc. cit.*

† Colladon, *Comptes Rendus de l'Académie des Sciences*, t. xv. p. 800, 1842.

RPA, RQA are equal to the angles of incidence and refraction, and

$$\sin RPA = \mu \sin RQA ;$$

therefore $RQ = \mu\, RP$.

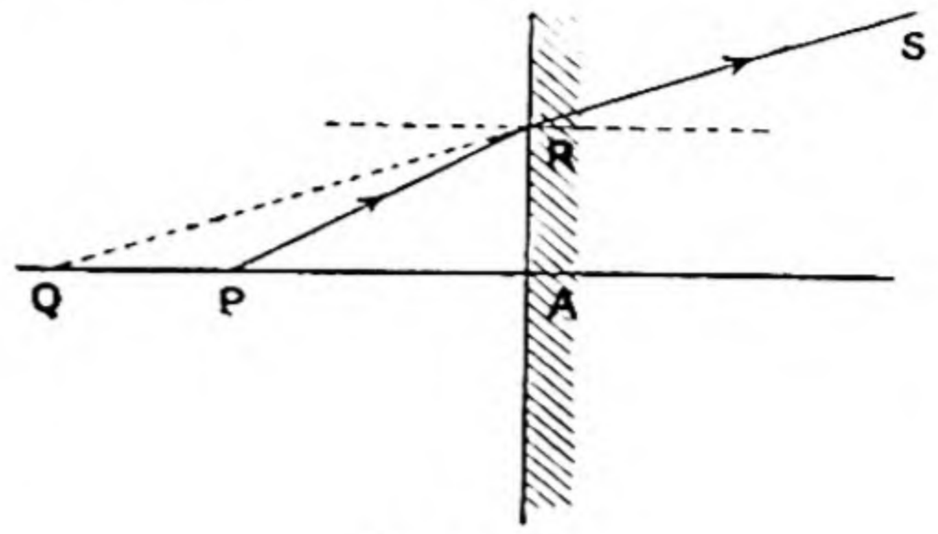


Fig. 23.

For rays at different inclinations to PA *the position of* Q *will vary*, but for rays which make small angles with PA, we have

$$RQ = AQ \text{ and } RP = AP, \text{ approximately,}$$

i.e. neglecting the squares of the circular measures of the inclinations of the rays to the normal.

Hence for rays which are incident nearly normally Q is a fixed point in relation to P, given by

$$AQ = \mu\, AP,$$

and in this sense we may call Q the image of P. But it must be remembered that it is *only pencils which make small angles with the normal that will diverge from a point after refraction.* Strictly speaking therefore, if rays diverging in all directions from a bright point are refracted by a plane surface, there is no image point through which the refracted rays pass.

33. ***An experiment for determining the refractive index of a rectangular block of glass.***

Place the block on a sheet of paper on a horizontal table. Make a scratch Q on one face of the block and stick a pin Pp in the table on the other side of the block so that the head of the pin is the same height above the table as the mark Q. Place the eye E in a convenient position slightly above the line QP and move the block to and fro until the image of P seen by reflection in the front surface of the glass appears to coincide at P' with the image of Q as seen by rays refracted at the same surface. P' is then the image of Q formed by refraction from glass to air, so that

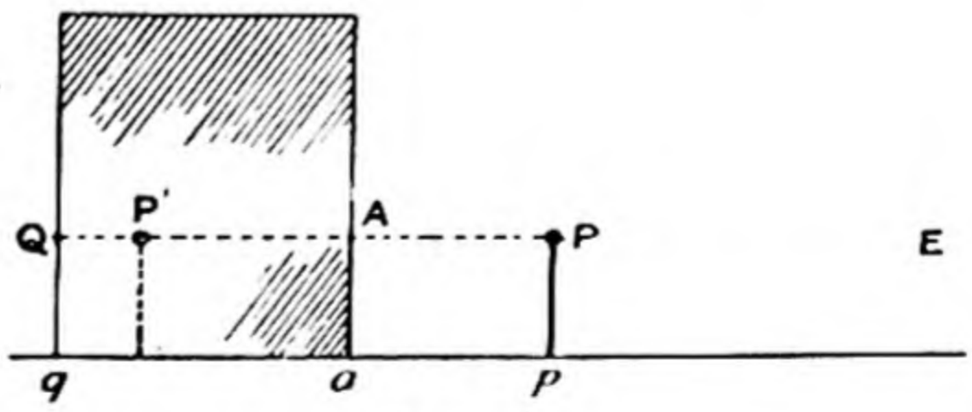


Fig. 24.

$$\mu = AQ/AP' = AQ/AP.$$

The distances AQ, AP can be measured and thus μ is determined.

34. *To shew that when rays proceeding from a point under water emerge obliquely and enter the eye, the rays that enter the eye all intersect two short straight lines at right angles to one another.*

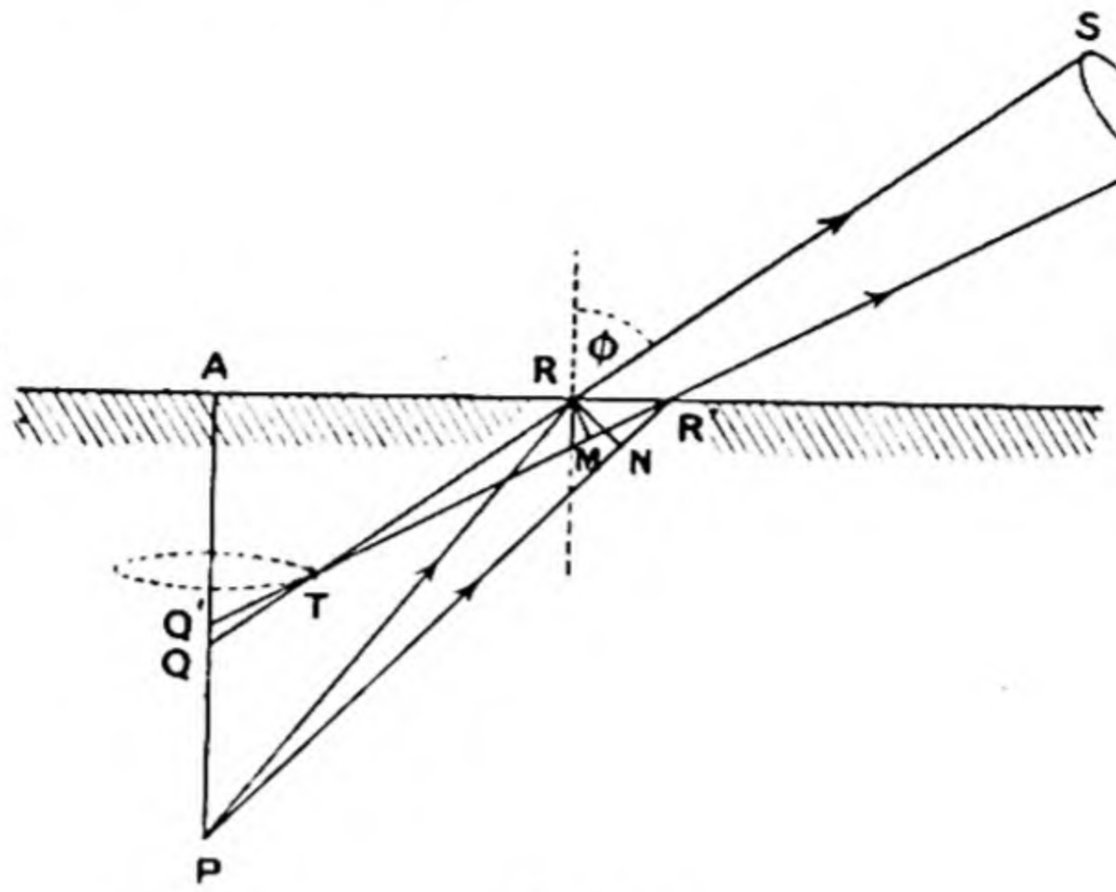


Fig. 25.

Let PRS, $PR'S'$ be two neighbouring rays that enter an eye at SS'.

Let SR, $S'R'$ cut the normal PA in Q, Q' and cut one another in T. It is clear that all rays in the plane of the paper between PRS and $PR'S'$ will intersect PA between Q and Q'.

If the figure is rotated about the line PA the rays describe cones about the axis PA with their vertices at P, Q, Q'. An eye at SS' will receive a bundle of rays lying between the two cones whose vertices are at Q and Q'; but these two cones intersect in the horizontal circle described by the point T when the figure is rotated about PA, therefore all the rays that enter the eye intersect a small arc of this circle; small because of the width of the pupil, and approximately a straight line. This line at T and the cross section of the pencil at Q which is approximately a straight line are called the **focal lines** of the pencil received by the eye.

The position of the focal lines may be found thus:

If ϕ, $\phi+\delta\phi$, ϕ', $\phi'+\delta\phi'$ are the angles made by the rays RS, $R'S'$, PR, PR' with the normal to the surface, then

$$AQ = AR \cot\phi = AP \tan\phi' \cot\phi,$$

where $\sin\phi = \mu \sin\phi'$. So that if AP and ϕ' are given, AQ is determined.

Again if RM, RN are at right angles to $R'S'$, PR',

$$TR\,\delta\phi = RM = RR' \cos\phi$$

and

$$PR\,\delta\phi' = RN = RR' \cos\phi';$$

therefore

$$TR = PR \cdot \frac{\cos\phi}{\cos\phi'} \frac{\delta\phi'}{\delta\phi}.$$

But

$$\cos\phi\,\delta\phi = \mu \cos\phi'\,\delta\phi';$$

therefore

$$TR = \frac{PR}{\mu} \frac{\cos^2\phi}{\cos^2\phi'}.$$

35. Image formed by refraction through a plane plate of glass.

Let μ be the refractive index and t the thickness of the plate. Let P be a point from which rays proceed striking the plate at small angles with the normal, P_1, P' the images formed after refraction into the glass at the surface A and out again at B. Then

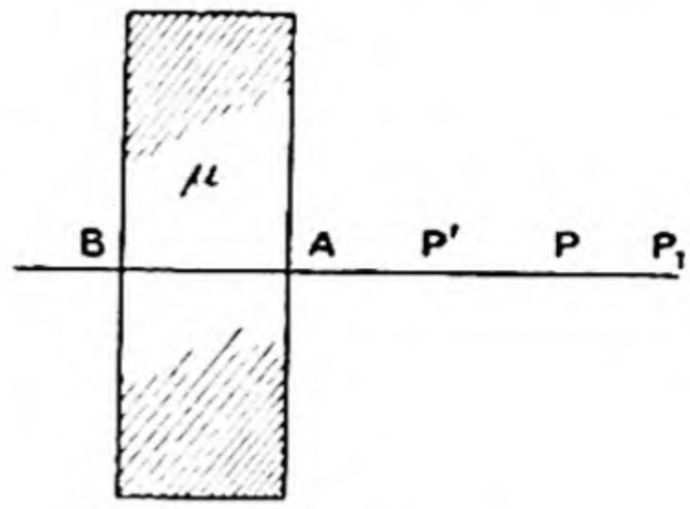


Fig. 26.

$$AP_1 = \mu AP,$$

and
$$BP' = \frac{1}{\mu} BP_1.$$

Hence to an eye on the other side of the plate the point P will appear to be a distance PP' nearer the plate than it really is, where

$$P'P = BP - BP' = t + AP - \frac{1}{\mu}(t + AP_1)$$
$$= t\left(1 - \frac{1}{\mu}\right).$$

That is the displacement depends only on μ and t and is independent of the position of P. Hence also the images of objects viewed directly through a glass plate are the same size as the objects.

36. The deviation produced by a single refraction increases with the angle of incidence and at an increasing rate.

By differentiating the equation

$$\sin\phi = \mu \sin\phi'$$

we get
$$\cos\phi \frac{d\phi}{d\phi'} = \mu \cos\phi';$$

therefore
$$\cos^2\phi \left(\frac{d\phi}{d\phi'}\right)^2 = \mu^2 - \sin^2\phi,$$

or
$$\left(\frac{d\phi}{d\phi'}\right)^2 = \mu^2 \sec^2\phi - \tan^2\phi$$
$$= (\mu^2 - 1)\sec^2\phi + 1 \quad \ldots\ldots\ldots\ldots(1).$$

Now assuming that $\phi > \phi'$, we must have $\mu > 1$, and therefore from the last equation $\frac{d\phi}{d\phi'} > 1$.

Hence if $D = \phi - \phi'$, $\qquad \dfrac{dD}{d\phi'} > 0$;

that is D increases with ϕ' or with ϕ.

Again, from (1) $\dfrac{d\phi}{d\phi'}$ increases with ϕ,

therefore $\qquad \dfrac{d(\phi - \phi')}{d\phi'}$ or $\dfrac{dD}{d\phi'}$ increases with ϕ;

that is D increases with ϕ at an increasing rate.

If the ray be passing from a denser to a rarer medium we have only to suppose the direction reversed and the argument still holds.

In applying this theorem to examples it should be noted that what has been proved is that if the sines of two variable acute angles are in a constant ratio, then the difference between the angles increases with either of them and at an increasing rate.

37. The theorem of the last article may also be proved geometrically*.

C is a point at distance μ from the centre O of a circle of unit radius.

If CPQ cuts the circle in P, Q and the angle $OCP = \phi'$; since

$$\sin OPQ : \sin OCP = OC : OP = \mu : 1,$$

therefore $\qquad \sin OPQ = \mu \sin \phi'$, and $OPQ = \phi$

and therefore $\qquad POC = \phi - \phi' = D.$

As the line CP revolves round C starting from the position CO the angle ϕ increases steadily from $0°$ to $90°$, the latter value corresponding to the position in which CP is a tangent to the circle; and it is obvious that D also increases steadily.

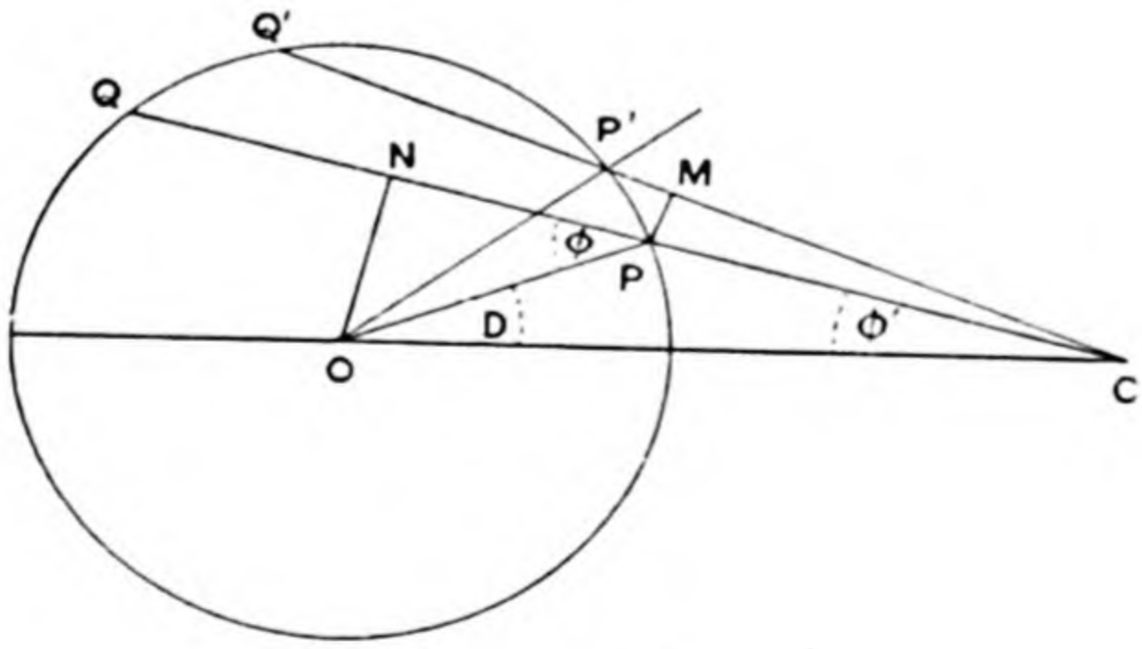


Fig. 27.

Again if $CP'Q'$ be a position near to CPQ, and $PCP' = \delta\phi'$, $POP' = \delta D$, and if PM be perpendicular to CP' we have

$$\delta D = PP' = PM \sec \phi = CP\, \delta\phi' \sec \phi.$$

* P. G. Tait, *Light*, Edinburgh, 1884, p. 90.

Hence if ON be perpendicular to CQ

$$\frac{dD}{d\phi'} = \frac{CP}{\cos\phi} = \frac{CP}{PN},$$

and this ratio clearly increases as ϕ' or ϕ increases. Hence D increases at an increasing rate.

38. Examples. 1. The sloping bank of a reservoir partly filled with water ($\mu = \frac{4}{3}$) is all in one plane, but when viewed from above is apparently bent upwards at the surface through an angle of $\tan^{-1}\frac{1}{7}$; find its inclination to the horizon, the image being formed by direct pencils. (Coll. Exam. 1887.)

2. A sheet of water 2 inches deep and with refractive index $\frac{4}{3}$ lies on a plane reflecting bottom. Two sources of light are placed 1 foot above the water and 3 inches apart from each other and are observed by an eye placed in the straight line joining them. If the image of one source formed after one reflection at the bottom appears to coincide with the image of the second source formed after two reflections at the bottom, shew that the rays of light inside the water make an angle $\sin^{-1}\frac{3}{5}$ with the normal to the surface and that the eye is 35 inches from the nearer source. (Coll. Exam. 1907.)

3. Shew that if a cube (edge of length a) be constructed of a substance whose index of refraction from air is $\frac{5}{3}$, a small object at the centre of the cube will not be visible if an opaque circular patch of radius $\frac{3}{8}a$ be placed at the centre of each face. (Coll. Exam. 1900.)

4. Describe the directions in which rays from outside would reach a fish inside a thin glass sphere full of water, when its distance from the centre is (i) less, (ii) greater than a/μ. Shew that, in the latter case, there is a certain space outside the sphere from which no ray can reach the fish. μ is the index of refraction from air to water, and a the radius of the sphere.
(Coll. Exam. 1909.)

5. A person holds a candle in front of a thick plate-glass window, explain why he sees several images of the candle and shew that their distances apart are unaltered by moving the candle. (M. T. 1902.)

6. A person looks into a mirror consisting of a plate of glass of thickness t with a silvered back. If his eye is at distance u from the front surface find the distance of its image behind the back surface. (Coll. Exam. 1913.)

7. An object viewed (by nearly normal rays) through a thick plate of glass seems to be $\frac{3}{8}$ of an inch nearer to the observer than it really is; how thick is the glass? (Take $\mu = 1{\cdot}5$.) (St John's Coll. 1906.)

8. From a point Q, whose depth beneath the surface of a refracting liquid equals h, a ray of light is incident on a plane vertical mirror sunk in the liquid, the plane of incidence being vertical, and the point of incidence in the same horizontal line with the geometrical focus of Q for direct incidence on the surface of the liquid; shew that if after reflection the ray can just emerge, and the distance of Q from the point of incidence on the mirror equals $h\tan a$, the index of refraction of the liquid equals $\sec 2a$.
(Coll. Exam. 1887.)

9. Rays incident on the curved surface of a hemisphere, in a direction perpendicular to its plane face, pass through the hemisphere. Shew that the deviation is greatest for the rays which fall most obliquely on the surface.
(Coll. Exam. 1903.)

39. Prisms. Any portion of a medium bounded by two plane faces which meet is called a *prism*, the angle between the planes is called the *refracting angle* of the prism. A plane perpendicular to the edge or line of intersection of the plane faces is called a *principal plane*.

40. When a ray of light passes through a prism which is optically denser than the surrounding medium the deviation is always away from the edge.

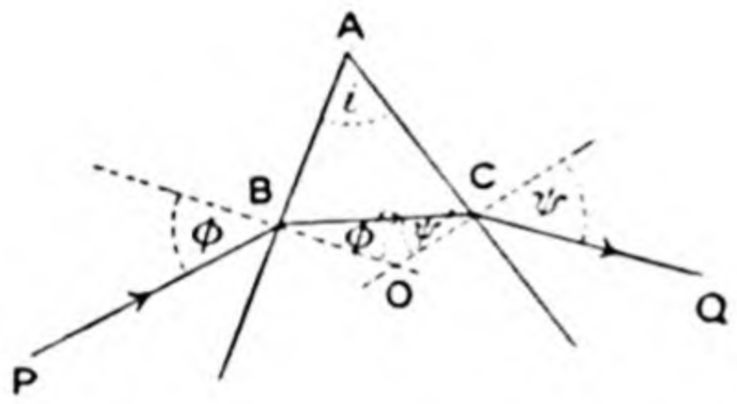


Fig. 28 (i).

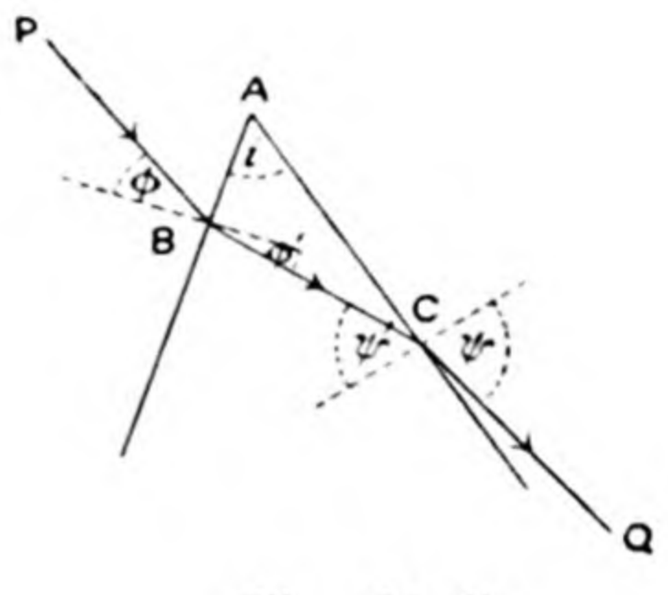


Fig. 28 (ii).

The path of the ray is $PBCQ$ and there are two cases to be considered—(i) when the path of the ray in the glass BC lies between both the normals at B and C and the edge A of the prism, in which case the ray is bent nearer to the normal at B at entrance and further from the normal at C on emergence, so that both refractions produce deviation away from the edge of the prism, as in Fig. 28 (i);

(ii) when the normal at one of the points B, C lies between the ray BC and the edge A of the prism, as in Fig. 28 (ii). Denoting the angles at B by ϕ, ϕ' and at C by ψ, ψ' as in the figure, it is easy to see from the triangle ABC (Fig. 28 (ii)) that

$$\psi' - \phi' = i \text{ the angle of the prism.}$$

The deviation at B is $\phi - \phi'$ towards the edge, and at C it is $\psi - \psi'$ away from the edge, but $\psi' > \phi'$, therefore

$$\psi - \psi' > \phi - \phi',$$

and on the whole the deviation is away from the edge.

41. The path of a ray through the prism.

In Fig. 28 (i) if the normals at BC meet in O, the points A, B, O, C are concyclic, whence it follows that

$$\phi' + \psi' = i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the angles are further connected by the relations

$$\left.\begin{aligned} \sin\phi &= \mu \sin\phi' \\ \sin\psi &= \mu \sin\psi' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and the deviation
$$\begin{aligned} D &= \phi - \phi' + \psi - \psi' \\ &= \phi + \psi - i. \end{aligned}$$

In Fig. 28 (ii) we have as before

$$\psi' - \phi' = i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)'$$

and
$$\begin{aligned} D &= \psi - \psi' - (\phi - \phi') \\ &= \psi - \phi - i, \end{aligned}$$

and equations (2) still hold.

From equations (1) and (1)′ it is clear that if the angle of the prism exceeds twice the critical angle either ϕ' or ψ' must exceed the critical angle and no ray of light can get through the prism.

42. The deviation is a minimum when a ray passes symmetrically through a prism.

We have $\sin\phi = \mu \sin\phi'$,

and $\sin\psi = \mu \sin\psi'$;

therefore by addition

$$\sin\tfrac{1}{2}(\phi + \psi)\cos\tfrac{1}{2}(\phi - \psi) = \mu \sin\tfrac{1}{2}(\phi' + \psi')\cos\tfrac{1}{2}(\phi' - \psi').$$

But $\phi + \psi - i = D$ and $\phi' + \psi' = i$,

therefore $$\sin\tfrac{1}{2}(D + i) = \mu \sin\tfrac{1}{2} i \frac{\cos\tfrac{1}{2}(\phi' - \psi')}{\cos\tfrac{1}{2}(\phi - \psi)}.$$

Hence when $\phi = \psi$ and $\phi' = \psi'$,

then $\sin\tfrac{1}{2}(D + i) = \mu \sin\tfrac{1}{2} i.$

But if $\phi > \psi$,

then $\phi - \phi' > \psi - \psi'$, (Art. 36)

or $\phi - \psi > \phi' - \psi'$,

so that $\cos\tfrac{1}{2}(\phi - \psi) < \cos\tfrac{1}{2}(\phi' - \psi')$,

and $\sin\tfrac{1}{2}(D + i) > \mu \sin\tfrac{1}{2} i.$

The same inequality can be shewn to hold if $\psi > \phi$.

Hence the minimum value of D is given by

$$\sin\tfrac{1}{2}(D + i) = \mu \sin\tfrac{1}{2} i,$$

and it occurs when $\phi = \psi$.

43. Use of concentric circles to construct the path of a ray through a prism in a principal plane.

Draw concentric circles of radii unity and μ. Draw a radius OA of the inner circle in the direction of the incident ray. From A draw a parallel to the normal to the surface on which the ray is incident meeting the outer circle in B. If BA makes angles ϕ, ϕ' with OA, OB as in the figure, then $\sin\phi = \mu \sin\phi'$ because $OB = \mu OA$, so that OB must be parallel to the path of the ray inside the prism. From B draw a line BC parallel to the second surface of the prism. Then the angle ABC is i, the angle of the prism, and $OBC = i \mp \phi'$, $-$ or $+$ according as BA, BC are on opposite sides or on the same side of OB. In either case OBC is ψ', the angle of incidence in the glass, and if BC meets the inner circle in C it makes with OC an angle ψ given by $\sin\psi = \mu \sin\psi'$, so that OC is the direction of the emergent ray. If BC does not meet the inner circle the ray does not emerge from the prism.

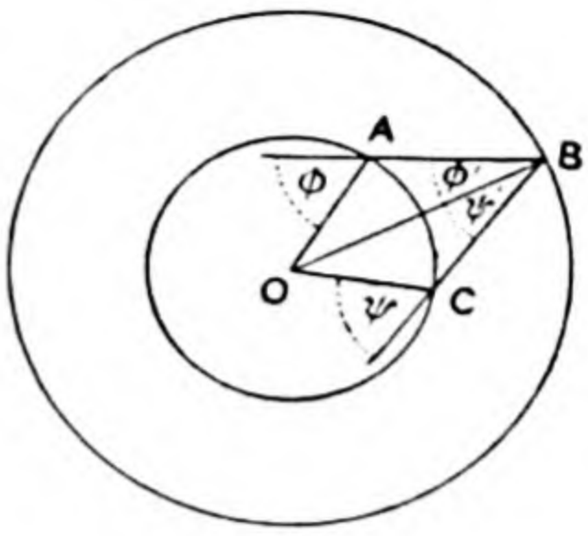


Fig. 29.

44. Field of view through a prism.

If α is the critical angle neither ϕ' nor ψ' can exceed α.

Hence, since $\phi' + \psi' = i$,

we have $\psi' \ngtr \alpha$,

and therefore $i - \phi' \ngtr \alpha$,

or $\phi' \nless i - \alpha$.

Therefore

$$\sin\phi \text{ or } \mu \sin\phi' \nless \mu \sin(i - \alpha),$$

or $$\phi \nless \sin^{-1}\{\mu \sin(i - \alpha)\}.$$

Similarly we must have

$$\psi \nless \sin^{-1}\{\mu \sin(i - \alpha)\}.$$

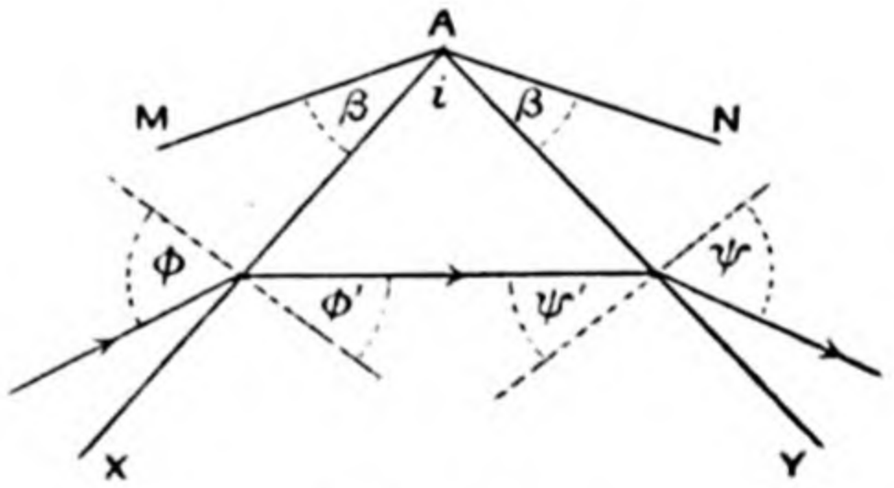


Fig. 30.

Calling this angle $90° - \beta$, we draw planes AM, AN through the edge A of the prism making angles $XAM = YAN = \beta$ with the faces AX, AY. Only objects within one of these angles will be visible on the other side of the prism and the eye that is to view the object through the prism must lie within the other angle.

45. Images.

In general rays diverging from a point and passing through a prism do not all pass through the same point on emergence, but if a small pencil of rays diverging from a point P passes with minimum deviation near the edge of a prism, the rays near to the

mean ray which passes symmetrically will make approximately the same angles with the mean ray on emergence as on incidence, and, as the mean ray is symmetrical with regard to the prism, if the emergent rays are produced backwards they will pass through a point P', and P, P' will be equidistant from the plane that bisects the angle of the prism.

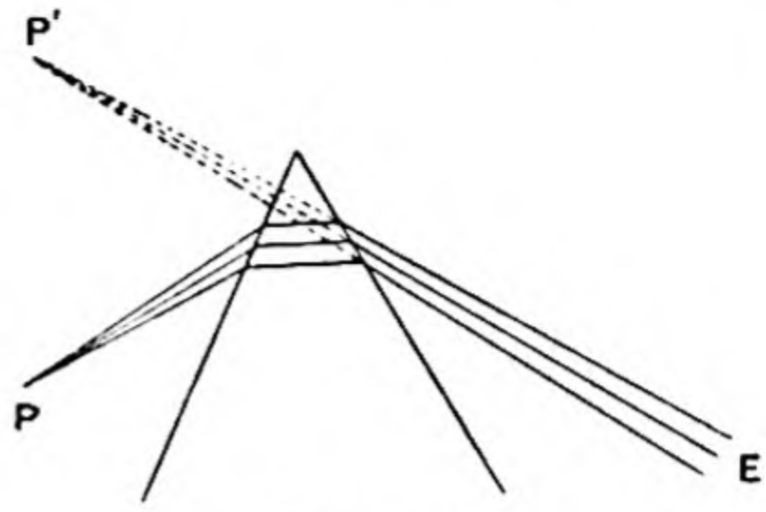


Fig. 31.

The same argument will hold for all the points of a small object at P viewed by an eye E with its centre on the mean ray, and the image at P' will appear to have the same size as the object would have if viewed directly at the same distance.

46. Examples. 1. Calculate the deviation produced in a ray by a glass prism of angle 60°, when the ray passes through symmetrically and $\mu = 1{\cdot}54$.
(St John's Coll. 1910.)

2. Calculate the deviation produced in a ray by a glass prism of angle 50° when the ray passes symmetrically and $\mu = 1{\cdot}54$. Calculate also the angular breadth of the field of view of the prism.
(Coll. Exam. 1911.)

3. The minimum deviation of a ray passing through a prism of angle 40° is found to be 32° 40′; find the refractive index of the medium of the prism.

Find also the deviation, if the angle of incidence of the ray is half what it is in the position of minimum deviation.
(Coll. Exam. 1908.)

4. The angle of a glass prism, refractive index $\frac{3}{2}$, is 60° and the angle of incidence is 45°; find the difference between the deviation in this case and the minimum deviation.
(M. T. i. 1913.)

5. Plot a graph to shew the deviation D at a single refraction as a function of the angle of incidence ϕ when the refraction is from water ($\mu = \frac{4}{3}$) into air.
(St John's Coll. 1911.)

6. Give a diagram shewing the possible course of rays in a principal plane after entering a prism of angle 60° at a given point—the prism being supposed to be optically more dense than the adjacent media and the critical angles at the surfaces of incidence and emergence being respectively 30° and 45°.
(Coll. Exam. 1913.)

7. Light is incident on one face of a prism, of which the refracting angle is 60°, in a plane perpendicular to the edge of the prism. The index of refraction of the glass is 1·5. Find the range of angles of incidence for which light is transmitted through the prism without internal reflection.
(M. T. i. 1908.)

8. Shew that no ray can pass through a prism whose angle i is greater than twice the critical angle ϵ. If $2\epsilon > i > \epsilon$, shew that if a ray whose angle of incidence is ϕ can pass through the prism then $\phi > \sin^{-1}(\sin i \cot \epsilon - \cos i)$.
(Coll. Exam. 1913.)

9. Shew that the angular extent of the field seen through an equilateral glass prism ($\mu = \frac{3}{2}$) is $\cos^{-1}\{(\sqrt{15}-2)/4\}$. (Coll. Exam. 1910.)

10. Shew that the angular breadth of the field of view in the principal plane of a prism of given angle is less the greater the refractive index of the material supposed optically denser than air. (M. T. i. 1909.)

11. The cross section of a prism is an equilateral triangle ABC. A ray is incident on AB, is internally reflected at AC and BC, and emerges from AB. Prove that the angle of incidence reckoned on the side of the normal remote from A must be greater than

$$\sin^{-1}\frac{\sqrt{(3\mu^2-3)}-1}{2}.$$

(Coll. Exam. 1904.)

12. The refractive index of a prism whose cross section is an equilateral triangle is 1·5. Find the limiting value of the angle of incidence when a ray passes through the prism in a principal plane without internal reflection.

Calculate the greatest deviation that can be produced in a ray of index 1·5 that is once totally reflected inside the prism. (M. T. ii. 1910.)

13. Two prisms are placed with their edges in the same direction and adjacent faces in contact; shew that if the angle of each prism exceed the critical angle for the medium of that prism, no ray can get through.
(Trinity Coll. 1891.)

14. Shew that the minimum deviation of a prism formed of a substance of given refractive index, continually increases with the angle of the prism, until that angle is so large that the ray can no longer pass through.
(St John's Coll. 1884.)

15. Prove that, if a prism is right-angled, and the least and greatest values of the deviation are respectively α and β,

$$\cos\alpha = \mu\cos\beta, \qquad \sin\alpha = \sin^2\beta.$$

Find $\cos\alpha$ and $\cos\beta$ when $\mu = 1{\cdot}4$. (M. T. i. 1912.)

16. A prism is placed between the eye and an object so that light from the object falls on the eye after refraction in a principal plane of the prism. The prism is placed, initially, so that the light falls on it normally, and it is then rotated slowly about an axis parallel to its refracting edge. Shew that the image appears first to approach the object and then to recede from it, and that the image is brightest when nearest the object. (M. T. i. 1910.)

17. ABC is a normal section of a triangular prism, whose refractive index is $\sqrt{2}$. The face AC of the prism is silvered, and the angle A is 60°. A ray of light, external to the prism and in the plane ABC, is incident at an angle of 45° at the foot of the perpendicular from A to BC. If 30° < angle C < 45°, prove that the direction of the ray on emerging from the prism is at right angles to its direction at incidence. (M. T. 1905.)

47. To measure the angle and determine the refractive index of a prism experimentally.

Place the prism with its edge A vertical on a sheet of paper on a horizontal table. Stick up a pin P opposite to the edge as in the figure. Place the eye in a convenient position to view the pin P by reflection on the line of the edge A and mark the line of sight by pins Q, Q'. Move to the other side of the prism and mark a similar line $R'RA$. If PA be produced to D, the equality of angles of incidence and reflection makes

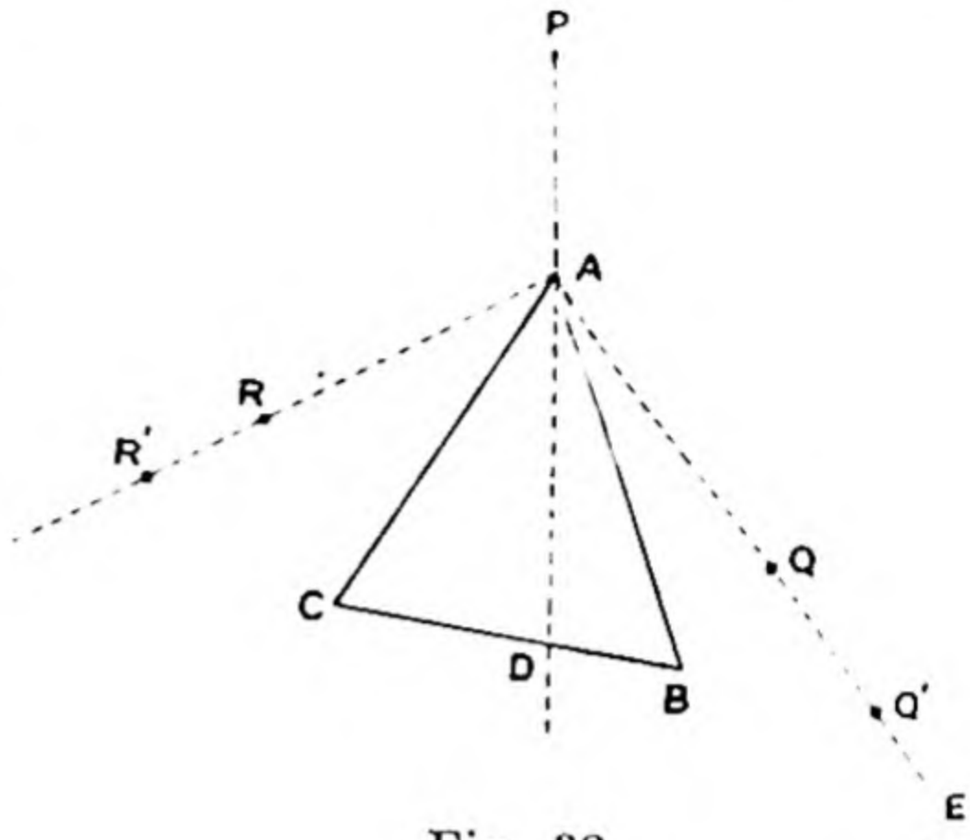


Fig. 32.

$$QAB=BAD \text{ and } RAC=CAD,$$
therefore $$BAC=\tfrac{1}{2}QAR,$$
and this angle can be measured.

To find the refractive index, fix the position of an incident ray by two pins P, P' and by viewing them through the prism the path of the emergent ray can be marked. Keeping PP' fixed, rotate the prism slowly and the deviation TOS will vary; find by adjustment the position which gives least deviation, and then measure the deviation D, and having previously found the angle i, the refractive index can be calculated by the formula for minimum deviation

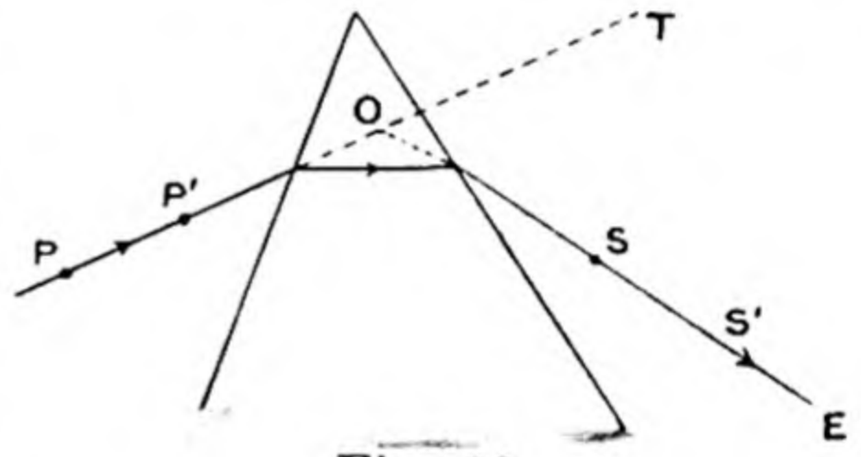


Fig. 33.

$$\sin\tfrac{1}{2}(D+i)=\mu\sin\tfrac{1}{2}i.$$

48. Refracted rays in more than one plane.

Three-dimensional problems can be solved by the help of the following propositions:

When a ray is refracted at the surface of separation of two media the angles which the incident and refracted rays make with any plane through the normal obey the law of refraction; and the projections of the rays on any plane through the normal obey a modified law of refraction.

Along the incident and refracted rays PO, OQ measure off lengths such that $PO=\mu$ and $OQ=\mu'$, where μ, μ' are the refractive indices of the two media. Let PM, QN be at right angles to the normal MON.

Since $$\mu\sin\phi=\mu'\sin\phi',$$
therefore $$MP=NQ.$$

If P', Q' are the projections of P, Q on any plane through MON, the angles $PP'M$, $QQ'N$ are right angles

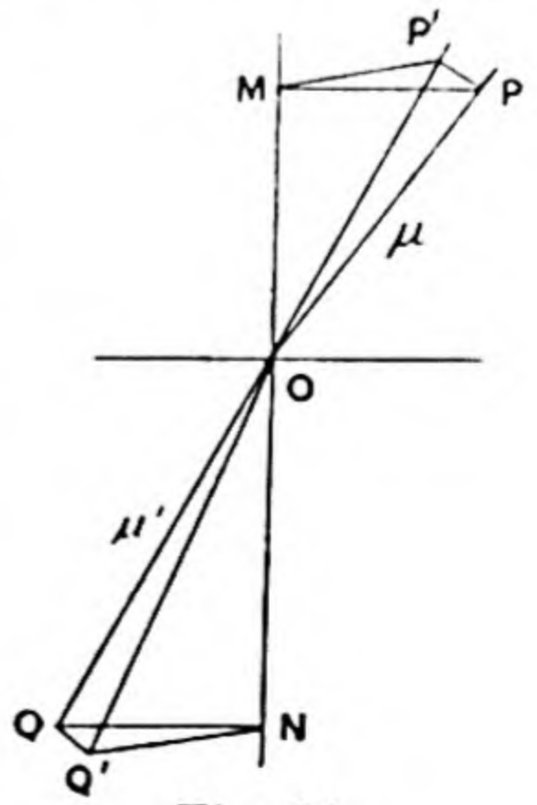


Fig. 34.

and the angles PMP', QNQ' are equal, being the angle between the two planes POQ, $P'OQ'$. Hence

$$PP' = QQ' \quad \text{and} \quad MP' = NQ'.$$

Now if a, a' are the inclinations of OP, OQ to the plane $P'OQ'$,

$$a = POP' \quad \text{and} \quad a' = QOQ',$$

so that $$PP' = \mu \sin a \quad \text{and} \quad QQ' = \mu' \sin a',$$

therefore $$\mu \sin a = \mu' \sin a'.$$

Again if $$P'OM = \theta \quad \text{and} \quad Q'ON = \theta',$$

$$OP' \sin \theta = MP' = NQ' = OQ' \sin \theta',$$

but $$OP' = \mu \cos a \quad \text{and} \quad OQ' = \mu' \cos a' ;$$

therefore $$\mu \cos a \sin \theta = \mu' \cos a' \sin \theta'.$$

49. Path of a ray through a prism not in a principal plane.

The method of Arts. 20, 43 may be applied as follows, using concentric spheres of radii, unity and μ. Draw a radius OA in the direction of the incident ray meeting the inner sphere in A. Let ON be parallel to the edge of the prism. A plane through A parallel to a principal plane cuts the spheres in two circles and meets ON at right angles at N, the common centre of the circles. Through A draw AB parallel to the normal to the surface on which the ray is incident, meeting the outer sphere in B, then since

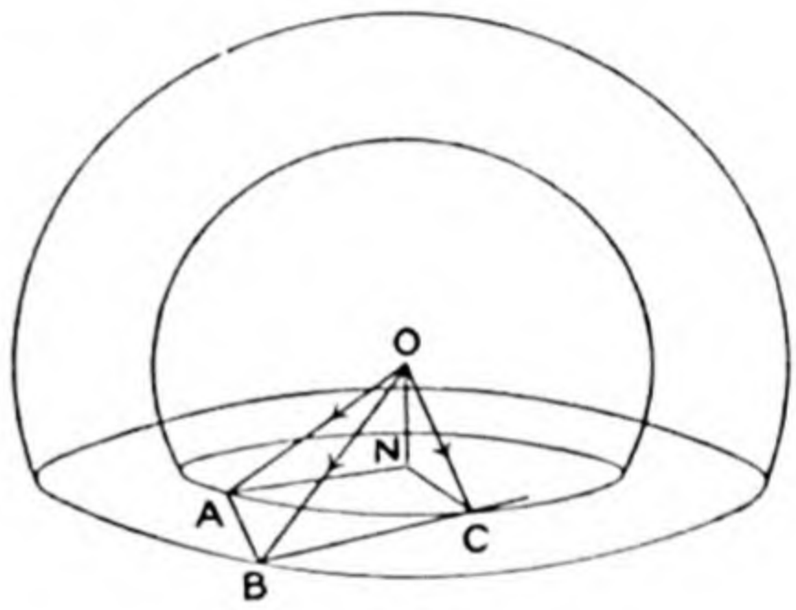


Fig. 35.

$$\sin OAB = \mu \sin OBA,$$

therefore OB is parallel to the ray in the prism.

Through B draw BC parallel to the normal to the second face of the prism meeting the inner sphere in C, then since

$$\sin OCB = \mu \sin OBC,$$

therefore OC is parallel to the emergent ray.

Since the normals to the faces are at right angles to the edge, the points A, B, C lie in the plane section at right angles to ON.

It is evident therefore that the incident and emergent rays are equally inclined to the edge of the prism.

Again NA, NB, NC are the directions of the projection of the path on a principal plane ; and if D be the actual deviation and D_0 the deviation in the projected path, $D = AOC$ and $D_0 = ANC$.

Therefore $$2AO \sin \tfrac{1}{2}D = AC = 2AN \sin \tfrac{1}{2}D_0,$$

but $$AN/AO = \sin AON = \sin \theta, \text{ say,}$$

where θ is the inclination of the incident ray to the edge of the prism.

Hence $$\sin \tfrac{1}{2}D = \sin \theta \sin \tfrac{1}{2}D_0.$$

EXAMPLES.

1. If a ray traverses a shell of glass bounded by concentric spheres, the inner and outer medium being air, shew that, assuming that the ray always enters the cavity, the deviation increases with the angle of incidence.

(M. T. 1903.)

2. Shew that, when a prism is more highly refractive than the surrounding medium, the greatest value of the minimum deviation, for varying angles of the prism, is $(\pi - 2\gamma)$, where γ is the critical angle for refraction from the surrounding medium into the prism. (Coll. Exam. 1902.)

3. Three right-angled prisms, of refractive indices μ_1, μ_2, μ_3, are fitted together so that the faces of the middle prism are in contact each with one of the faces of the outside prisms, the edge of the middle one being in the contrary direction to the edges of the other two. Shew that any ray which passes through all three prisms in a principal plane will pass without deviation if

$$\mu_1^2 + \mu_3^2 = 1 + \mu_2^2.$$

(Coll. Exam. 1894.)

4. A pencil of parallel rays strikes a face of a triangular prism in a direction perpendicular to the edges and parallel to a second face. It suffers total reflection at the second face and emerges after refraction at the third: find the condition that the deviation may be zero. (Coll. Exam. 1905.)

5. If a prism of angle $2a$, made of glass whose critical angle is γ, has a plane base, so as to be isosceles, and be laid with its base on a printed page, shew that an eye looking into the prism in a direction making with the base of the prism an angle less than

$$a + \sin^{-1}(\cos a + \gamma / \sin \gamma)$$

cannot see the print.

(Coll. Exam. 1905.)

6. $ABCD$, $DCEF$ are cross sections, in a principal plane, of two square prisms (refractive indices μ_1 and μ_2, where $\mu_1 > \mu_2$). A ray is refracted at the faces AD, DC and CE, the first angle of incidence being ϕ_1 and the last angle of refraction being ϕ_2; shew that $\sin^2 \phi_1$ must be greater than $\mu_1^2 - \mu_2^2$ and that for a minimum deviation ϕ_1 and ϕ_2 are complementary. (Coll. Exam. 1908.)

7. A ray is incident in any direction, not necessarily in a principal plane, on a prism of glass whose cross section is a regular polygon with an even number of sides, and after refraction it is reflected at all the other faces in succession and then emerges through the first face. Prove that its final direction is the same as if it were simply reflected at the first face.

(M. T. 1904.)

8. A person holds a candle in front of the bevelled edge of a mirror on the perpendicular to the reflecting surface through the intersection of the flat and bevelled surfaces of the glass; if the angle between the latter surfaces is $180° - i$, where i is very small, and the distance x of the candle from the glass is such that the incidence may be considered direct, shew that he can see two

bright images of the candle and that the distance between them parallel to the mirror is $2i(\mu-1)\left(x+\frac{t}{\mu}\right)$, where t is the thickness of the sheet of glass, and powers of i above the first are neglected.

Explain why other less distinct images appear on the bevelled surface as the candle is made to approach the glass. (Coll. Exam. 1899.)

9. Prove that if a refracting surface is such that all rays parallel to a certain line, which are incident on it, are refracted to the same point, the surface must be a prolate spheroid, whose eccentricity = the reciprocal of the index of refraction, μ, provided that μ is > 1. (Coll. Exam. 1905.)

10. A conical pencil of light of angular breadth 2θ from a vertex A is incident directly on one face of an isosceles rectangular prism, suffers total reflection at the base, emerges from the second face, and illuminates a screen parallel to the second face at any distance from it. Prove that the breadth of the bright part of the screen, in the principal plane through the axis of the pencil, is less than the breadth which the pencil in the absence of the prism would have at a distance from A equal to the length of the path of the axial ray from A to the screen, by

$$2c\tan\theta\left\{1-\frac{\cos\theta}{\sqrt{(\mu^2-\sin^2\theta)}}\right\},$$

where c is the length in the principal plane of a face of the prism.

(M. T. 1901.)

11. Shew that the deviation of a beam of light by a prism *of small angle* is approximately independent of the angle of incidence ϕ of the beam, when that angle is small.

Shew that when that angle is not small, the deviation is approximately equal to the angle of the prism multiplied by

$$\{(\mu^2-1)\sec^2\phi+1\}^{\frac{1}{2}}-1.$$

(St John's Coll. 1897.)

12. A prism of refractive index μ is placed between two media of refractive indices μ_1 and μ_2; prove that for minimum deviation the angle of incidence (ϕ) is connected with the angle of emergence (ψ) by the equation

$$\mu_1^2(\mu_2^2-\mu^2)\sin^2\phi-\mu_2^2(\mu_1^2-\mu^2)\sin^2\psi+\mu^2(\mu_1^2-\mu_2^2)=0.$$

(Trinity Coll. 1904.)

13. A prism of refractive index μ separates a region of index μ_1 from a region of index μ_2. A ray of light passes from the μ_1 region through the prism into the μ_2 region. If ϕ' and ψ' be the angles which the ray, when in the prism, makes with the normals to the first and second faces of the prism respectively, prove that

$$(\mu^2-\mu_1^2)\cos^2\psi'=(\mu^2-\mu_2^2)\cos^2\phi',$$

for the minimum deviation of the ray. (Coll. Exam. 1903.)

14. A ray is incident, in a plane perpendicular to the edges, on a face of a prism whose section is a regular polygon of $2n$ sides, and after internal reflection at each of the other faces emerges at the same face as before. Prove that the deviation is $\pi+2\phi$, where ϕ is the angle of incidence.

(Coll. Exam. 1896.)

15. Two prisms with refracting angles a_1, a_2 and indices μ_1, μ_2 are placed with their faces in contact, the refracting edges being turned opposite ways. Shew that if the deviation is a minimum when the ray passes perpendicularly from one prism to the other, then

$$\sin^2 a_1 (1 - \mu_1^2 \cos^2 a_2) = \sin^2 a_2 (1 - \mu_2^2 \cos^2 a_1).$$

(Coll. Exam. 1909.)

16. A ray is incident on a prism so that it is internally reflected at the second face; shew that the number of internal reflections which it must undergo at the two faces in succession cannot be more than the integer next greater than $\left(\frac{\pi}{2} - a\right) \Big/ i$, where i is the angle of the prism, and a is the critical angle of the material. (Trinity Coll. 1890.)

17. Shew that a ray of light emerges from a prism at the same inclination to the refracting edge as it enters it: and find the minimum deviation for a ray whose inclination to the edge is given. (St John's Coll. 1896.)

18. A ray passes through a prism, not necessarily in a principal plane. Shew that the external bisector of the angle between the incident and emergent rays is perpendicular to the edge. (St John's Coll. 1904.)

19. Adapt the construction representing the directions of a ray refracted through a prism not in a principal plane to the case of a ray reflected at two plane mirrors and prove that the deviation is $2 \sin^{-1} (\sin a \sin i)$, where a is the angle the incident ray makes with the line of intersection of the mirrors, i the angle between the mirrors. (Coll. Exam. 1892.)

20. A ray is incident at an angle $2 \cos^{-1} \frac{1}{2}\mu$ on one face of a prism of refractive index μ, and after refraction through the prism it emerges at the same angle. Prove that the total deflection of the ray is equal to the refracting angle i, and that in the course of its passage through the prism the ray is inclined at an angle

$$\cos^{-1} \{\tfrac{1}{2}\mu \sec \tfrac{1}{2}i\}$$

to the principal plane. (St John's Coll. 1895.)

21. Prove that when a ray of light passes through a prism in a principal plane

$$\frac{\sin^2 (\phi - \phi')}{\sin^2 \phi'} + \frac{\sin^2 (\psi - \psi')}{\sin^2 \psi'} - \frac{2 \sin (\phi - \phi') \sin (\psi - \psi')}{\sin \phi' \sin \psi'} \cos i = 2 (1 - \cos D);$$

and shew that the result holds also when the ray does not lie in a principal plane. (Coll. Exam. 1902.)

22. A ray is refracted in any manner through a number of homogeneous media bounded by circular coaxial cylinders. Prove that the perpendicular distance between any portion of the ray and the axis is proportional to the tangent of the angle which it makes with the axis. (Coll. Exam. 1895.)

CHAPTER IV

REFLECTION AND REFRACTION AT SPHERICAL SURFACES

50. Definitions and Conventions.

Let a pencil of rays diverge from a point Q and fall on a spherical surface of centre O. The line QO is called the **axis** of the surface and the point A in which it meets the sphere is called the **vertex**. In general in what follows we shall only consider rays which make with the axis angles so small that the squares of their circular measures may be neglected. The incidence of such rays on the surface is said to be **direct**.

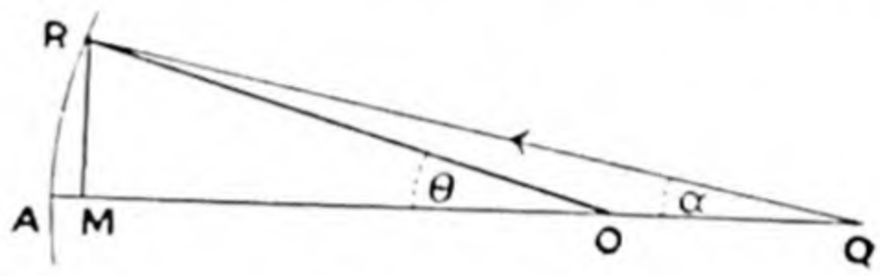


Fig. 36.

If QR be such a ray and $RQA = \alpha$ is its inclination to the axis, α is called the **divergence** of the ray.

Let $RM = y$ be the ordinate of the point R, let r be the radius and let OR make an angle θ with the axis, then θ is of the same order as α and its square can be neglected.

Hence $AM = AO\,(1 - \cos\theta) = 0$ to the first order

and $$y = RM = AO \sin\theta = r\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \text{arc } AR \quad \ldots\ldots\ldots\ldots\ldots\ldots$$

Also $$RQ = MQ \sec\alpha = AQ \quad \ldots\ldots\ldots\ldots\ldots\ldots$$

In what follows we shall use AB to denote a length in a geometrical sense, *i.e.* $BA = -AB$; we shall regard the light as coming from right to left in the standard case and lines drawn from left to right, *i.e.* to meet the incident light, as positive. With this convention, in the above figure AO, AQ, OQ are positive and OA, QO, QA are negative.

51. Reflection at a spherical mirror.

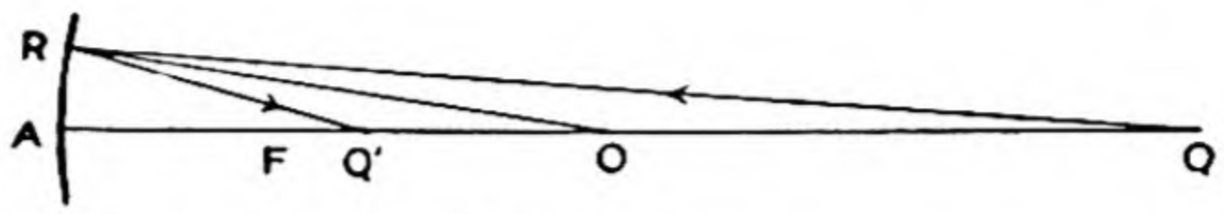


Fig. 37 (i).

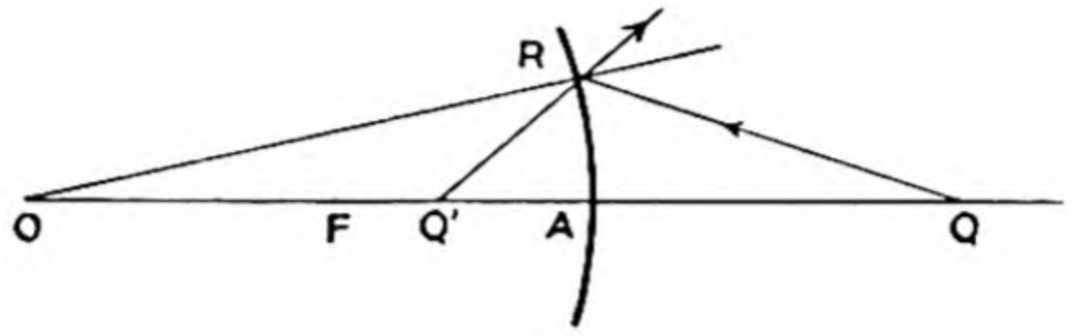


Fig. 37 (ii).

With the notation of the last Art. let the ray QR be reflected and cut the axis again in Q'. Then OR bisects the angle between the incident and reflected rays. In the first figure the mirror is concave and in the second convex to the incident light. The figures are lettered so as to correspond and if due regard is paid to the convention of signs any formula proved for either figure will also hold for the other. Hence in proving a formula connecting distances measured from the vertex A it is convenient, though not essential, to look at the first figure in which all lines measured from A are positive, and in proving a formula connecting distances measured from O it is convenient to look at the second figure in which all distances measured from O are positive.

Because OR bisects the angle between QR, $Q'R$,

therefore
$$Q'O : OQ = RQ' : RQ$$
$$= AQ' : AQ.$$

Hence
$$Q'O \,.\, AQ = OQ \,.\, AQ' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If we express this in terms of distances measured from A we have

$$(AO - AQ')\, AQ = (AQ - AO)\, AQ',$$

or, dividing by the product $AO \,.\, AQ \,.\, AQ'$,

$$\frac{1}{AQ'} + \frac{1}{AQ} = \frac{2}{AO} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If we denote the distances AQ, AQ', AO by u, v, r this result may be written

$$\frac{1}{v} + \frac{1}{u} = \frac{2}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3);$$

and in using this formula it is to be remembered that in the second figure v and r represent negative lengths.

Again (1) may be written

$$-OQ' \,.\, AQ = -OQ \,.\, Q'A,$$

or
$$OQ'\,(OQ - OA) = OQ\,(OA - OQ').$$

By dividing by $OQ' \,.\, OQ \,.\, OA$ this becomes

$$\frac{1}{OQ'} + \frac{1}{OQ} = \frac{2}{OA} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4);$$

or, denoting the distances OQ, OQ', OA by p, q, r we have

$$\frac{1}{q} + \frac{1}{p} = \frac{2}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

In using this formula it is to be remembered that in the first figure q and r represent negative lengths.

Either of the formulae (2), (4) expresses the fact that $(AQ'OQ)$ is a harmonic range*, from which it follows that if F is the middle point of AO,

$$FQ \,.\, FQ' = FO^2 = FA^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

52. Geometrical Focus.

Since Q' is a definite point whose position as determined by (2), (4) or (6) depends only on the position of Q and not on any particular ray, and since all rays from Q, making small angles with the axis within the meaning of Art. 50, after reflection pass through Q', Q' is called the **geometrical focus** of Q; and Q, Q' are called **conjugate points** or **conjugate foci.**

* The harmonic property is evident directly from the consideration that R is near enough to A for the angle ORA to be regarded as approximately a right angle, so that RO, RA bisect internally and externally or *vice versa* the angles of the triangle $Q'RQ$ and therefore $(AQ'OQ)$ is harmonic and $\frac{1}{AQ'} + \frac{1}{AQ} = \frac{2}{AO}$ as before.

The result (6) may be deduced from (1) thus :-

We have
$$\frac{AQ'}{Q'O} = \frac{AQ}{OQ},$$

so that
$$\frac{AQ' + Q'O}{AQ' - Q'O} = \frac{AQ + OQ}{AQ - OQ}$$

or
$$\frac{2FO}{2FQ'} = \frac{2FQ}{2FO}, \quad i.e. \quad FQ \,.\, FQ' = FO^2 = FA^2.$$

53. Images and Magnification.

We have seen that to our order of approximation every ray that diverges from Q will after reflection pass through a point Q' on QO, and in the same way every ray that diverges from P will after reflection pass through P' on PO; and if $OP = OQ$ then $OP' = OQ'$.

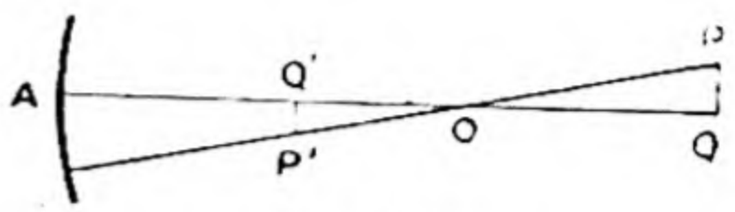


Fig. 38.

Now when P is sufficiently near to Q the line PQ is approximately at right angles to the axis AO and in like manner $P'Q'$ is at right angles to AO, and if we suppose the figure rotated round the axis AO the lines PQ, $P'Q'$ trace out small plane areas at right angles to the axis and to our order of approximation every ray proceeding from a point on the area at Q will after reflection pass through a point on the area at Q', so the latter is called the **image** of the former.

The ratio $P'Q' : PQ$ is the ratio of the linear dimensions of the image and object at right angles to the axis and is called the **linear magnification.**

The magnification is regarded as negative when the image is an inverted one. Denoting the linear magnification by m we have

$$m = \frac{P'Q'}{PQ} = \frac{OQ'}{OQ} = \frac{q}{p} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1);$$

also, since $$\frac{OQ'}{OQ} = \frac{Q'A}{AQ} = -\frac{v}{u} \quad \text{(Art. 51)},$$

therefore $$m = -\frac{v}{u} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

54. Longitudinal Magnification.

If we regard the object as possessing thickness denoted by δu in the direction of the axis, the corresponding thickness of the image is δv where the relation between δv and δu is got by making small variations in v and u in the formula

$$\frac{1}{v} + \frac{1}{u} = \frac{2}{r}.$$

Thus we have $$\frac{\delta v}{v^2} + \frac{\delta u}{u^2} = 0.$$

So if we denote longitudinal magnification by m' we have

$$m' = \frac{\delta v}{\delta u} = -\frac{v^2}{u^2} = -m^2.$$

55. Principal Focus.

A small pencil of rays incident parallel to the axis will after reflection pass through F the middle point of AO. For in this case the angle

$$QRO = ROF,$$

therefore $$FRO = ROF,$$

and $$FO = RF = AF,$$

to our order of approximation.

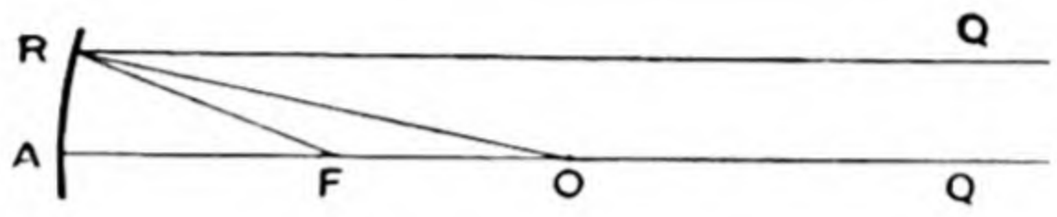


Fig. 39.

This is otherwise obvious from the formulae of Art. 51. Thus if in (3) we make $u \to \infty$ we get $v = \frac{1}{2}r$ or Q' coincides with F. In the same way if in (5) we make $p \to \infty$ we get $q = \frac{1}{2}r$ or Q' coincides with F. Also if in (6) FQ increases to infinity, FQ' must decrease to zero, or Q' moves up to F.

The point F is called the **principal focus** of the mirror.

Since the path of a ray is reversible it is clear that if rays diverge from F they will be reflected parallel to the axes. This also follows from the formulae (3), (5) which shew that if Q moves up to F so that $u = p = \frac{1}{2}r$, v and q must be infinite, and from (6) if Q moves up to F, Q' must move away to infinity.

From equation (6) we also learn that, since the product $FQ.FQ'$ is positive, a pair of conjugate points lie on the same side of the principal focus.

56. A plane through the principal focus at right angles to the axis is called the **principal focal plane** of the mirror; from Art. 53, it is clear that any small pencil of parallel rays making a small angle with the axis will after reflection converge to or diverge from some point in the principal focal plane.

This point may be found as in Fig. 40 by taking that one PA of the parallel rays that strikes the mirror at the vertex A and making the angle $FAP' = FAP$. If AP' meets the principal focal plane in P' then the parallel rays after reflection pass through P'.

Otherwise, it is obvious that the ray which passes at incidence through O the centre of the mirror will retrace its path after reflection, therefore the ray through O cuts the focal plane in the required point P'.

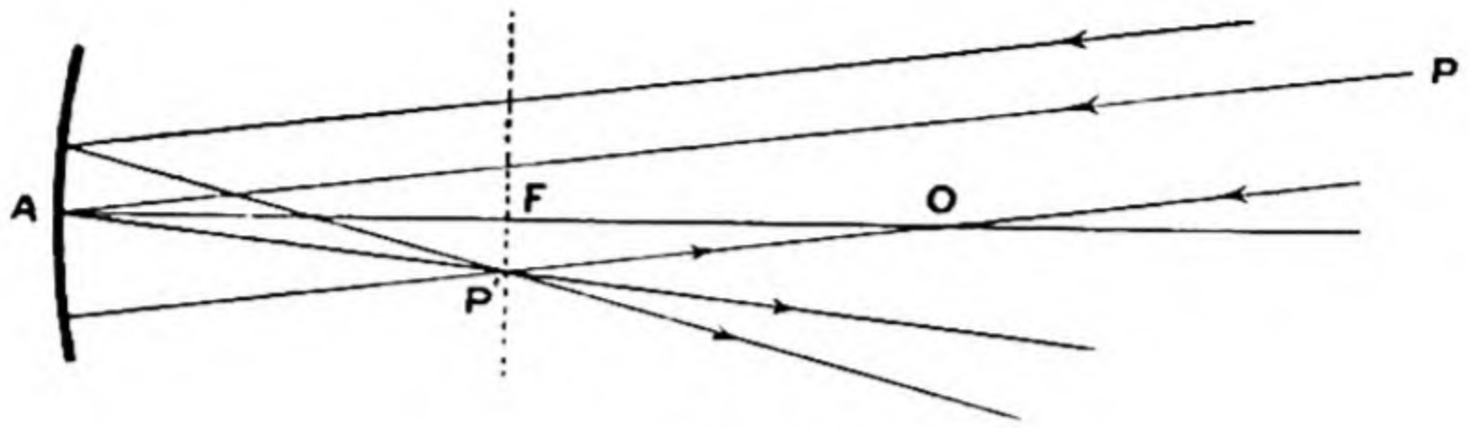


Fig. 40.

The length AF is called the **focal length** of the mirror; it is positive for a mirror concave to the incident light and negative for a mirror convex to the incident light.

When the terms "focus" or "focal plane" of a mirror are used they are understood to mean *principal* focus and *principal* focal plane.

57. To construct the image of a small object.

Let the line PQ at right angles to the axis denote a small object situated at Q.

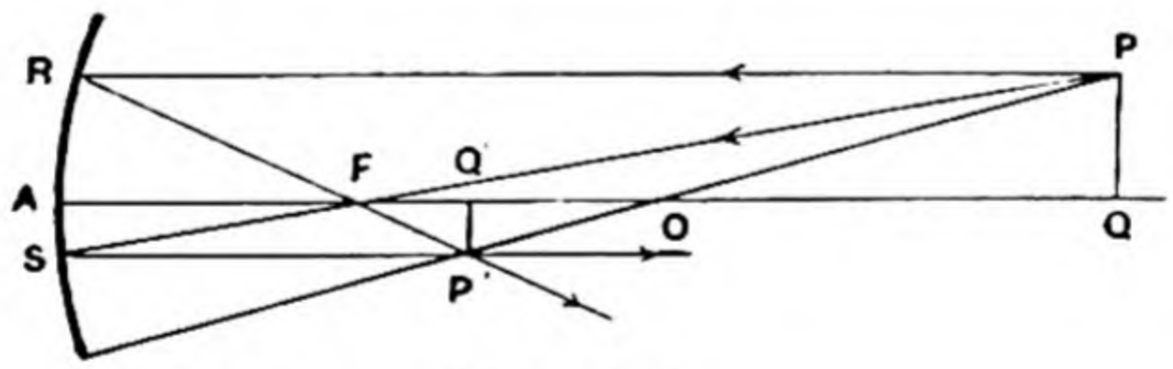


Fig. 41 (i).

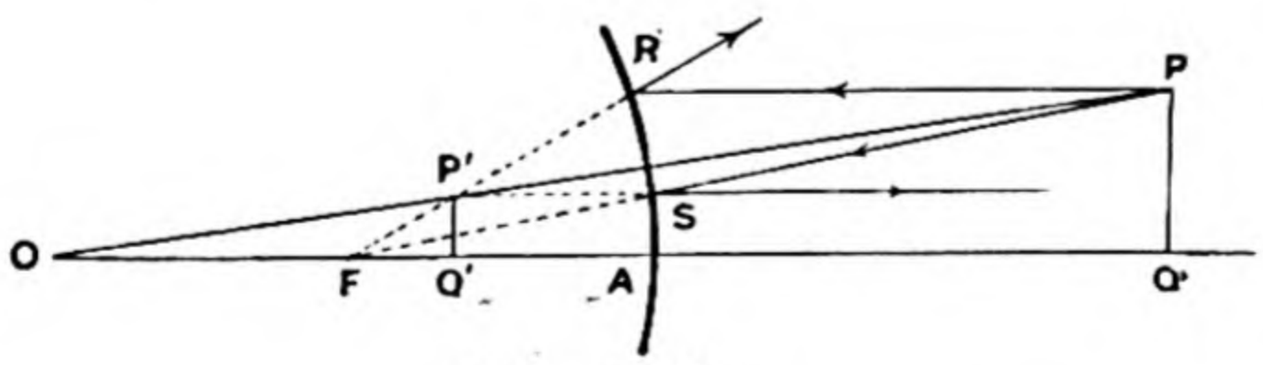


Fig. 41 (ii).

If a ray from P parallel to the axis meets the mirror in R its path after reflection is the line RF where F is the focus.

Also a ray PS incident in the direction PF is reflected parallel to the axis. Thus we get the positions of two reflected rays, and their intersection P' must be the image of P, and $P'Q'$ perpendicular to the axis is the image of PQ.

We have a further check that a ray PO, where O is the centre of the mirror, will retrace its path after reflection and therefore P' must lie on PO.

From the figures we can deduce further expressions for the linear magnification, and verify formulae already established.

Thus we have

$$m = \frac{P'Q'}{PQ} = \frac{SA}{PQ} = \frac{FA}{FQ},$$

also

$$= \frac{P'Q'}{RA} = \frac{FQ'}{FA}.$$

This means that the linear magnification varies inversely as the distance of the object or directly as the distance of the image from the focus of the mirror.

Generally speaking these formulae for m are more convenient to use than q/p or $-v/u$, because they only involve the position of object *or* image, whereas the use of q/p and $-v/u$ involve knowing the position of both object and image.

From the formulae just obtained we see that

$$FQ \,.\, FQ' = FA^2,$$

as in Art. 51.

58. To examine the changes in the magnitude and position of the image as a small object approaches a concave mirror.

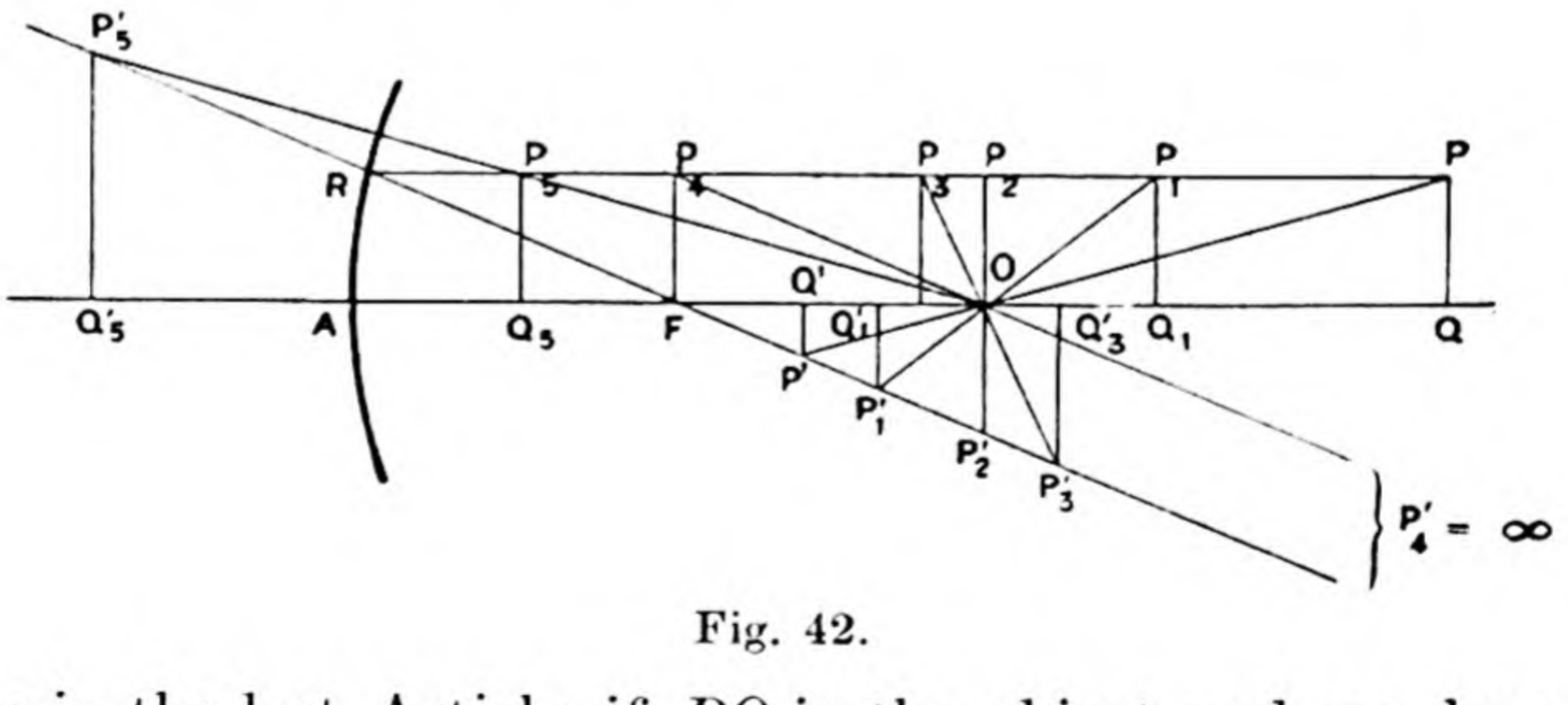


Fig. 42.

As in the last Article, if PQ is the object and we draw a ray PR parallel to the axis, the image P' of P lies on the line RF and also on the line PO.

As PQ approaches the mirror the line RF is unchanged in position but the line POP' turns round the point O. The figure shews a number of positions and makes it clear that

(1) so long as the distance of PQ from the mirror is greater than the radius the image $P'Q'$ is real, inverted and smaller than the object;

(2) when PQ is at the centre O, the image is real, inverted and equal to the object;

(3) when PQ lies between O and F, the image is real, inverted and larger than the object;

(4) as the object approaches the focus the image recedes from it, increasing in magnitude;

(5) when PQ is at F, the image is infinite in size and at an infinite distance;

(6) as soon as PQ passes F, the image becomes a *virtual* erect image decreasing in size as PQ moves up to the mirror, and becoming equal to and coincident with the object when the latter reaches the mirror.

These changes are easily seen by looking into a concave mirror. At a distance from the mirror an inverted image of the face of the observer is seen and as the observer approaches the focus the image becomes too large for the part of it within the field of view to be distinguishable; then as soon as the focus is passed the image suddenly changes to an erect one which diminishes continually until the mirror is reached.

The case in which the incident rays are converging towards a point P *behind* the mirror may be considered in like manner, the object in this case being "virtual" and the image lying between the mirror and the focus.

The same problem for a convex mirror may be treated similarly, and it appears that in a convex mirror the image of a real object is always erect and virtual, but rays converging to a point behind the mirror, *i.e.* to a "virtual object," will have a real erect image if the virtual object lie between the focus and the mirror, and a virtual inverted image if the virtual object lie beyond the focus of the mirror.

59. Examples. 1. A candle is placed 7 inches in front of a concave spherical mirror of radius 1 foot. Find the position of the image.

2. Where must a small object be placed in relation to a concave spherical mirror of radius 2 feet in order that an image magnified three times may be formed? Shew that there are two solutions according as the image is erect or inverted. Are the images real or virtual?

3. If the focal length of a concave mirror be 15 ins. find the position and height of the image of an object 3 ins. high placed 12 ins. from the mirror. Is the image real or virtual?

4. Find the position and magnitude of the image of a disc 3 ins. in diameter placed at right angles to the axis of a convex mirror of radius 6 feet, according as the distance of the disc from the mirror is (1) 1 foot, (2) 3 feet, (3) 9 feet.

5. Work the same question, when the mirror is concave.

6. An object 5 cms. high is placed at a distance of 10 cms. from a spherical mirror, and a virtual image 1 cm. high is formed. Find the radius of curvature of the mirror. Is it concave or convex?

7. A man walks up to a spherical mirror, until his face is close to it. Describe the changes in the reflection he sees of himself, both when the mirror is convex and when it is concave. (Coll. Exam. 1907.)

8. Assuming that the sun's diameter subtends at the earth an angle of half a degree, calculate the approximate size of the image of the sun formed by a concave spherical mirror of 2 ft. radius. (St John's Coll. 1906.)

60. Refraction at a spherical interface.

Let QRS be the path of a ray passing from a medium of refractive index μ to one of refractive index μ'.

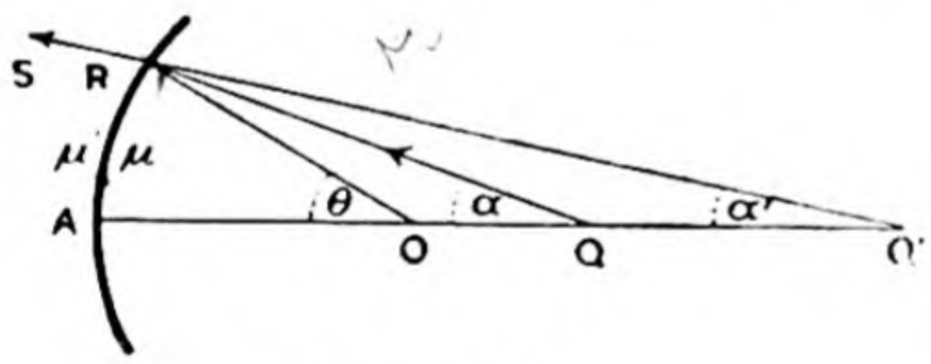


Fig. 43 (i).

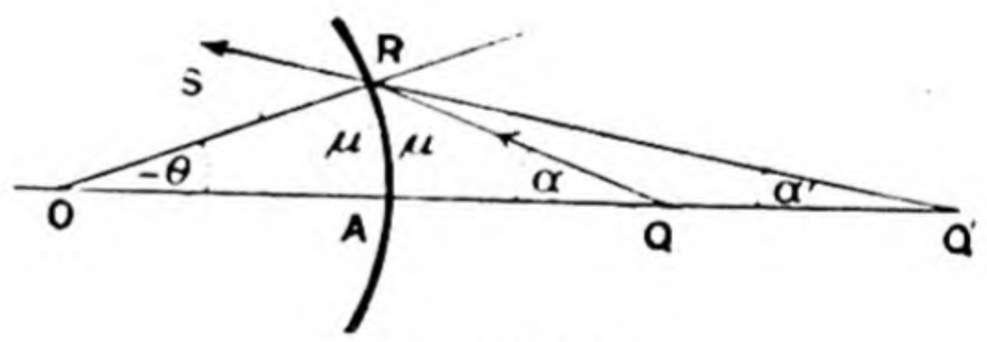


Fig. 43 (ii).

Let the refracted ray RS meet the axis AOQ in Q', let α, α' be the angles of divergence of the rays, *i.e.* the angles RQA, $RQ'A$, and let $ROA = \theta$ positive when, as in the first figure, it is measured in the same sense as α regarded as positive, and negative when as in the second figure.

Then $$\mu \sin(\theta - \alpha) = \mu' \sin(\theta - \alpha'),$$

or, to our order of approximation,

$$\mu(\theta - \alpha) = \mu'(\theta - \alpha');$$

therefore $$\mu'\alpha' - \mu\alpha = (\mu' - \mu)\theta = \frac{\mu' - \mu}{r} y \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where r is the radius AO and y is the ordinate at the point R.

The expression $(\mu' - \mu)/r$ is called the **power of the refracting surface**; it is positive if $\mu' > \mu$ and the surface is concave to the

incident light, or if $\mu' < \mu$ and the surface is convex to the incident light.

Since $\alpha' = y/AQ'$ and $\alpha = y/AQ$, equation (1) may be written

$$\frac{\mu'}{AQ'} - \frac{\mu}{AQ} = \frac{\mu' - \mu}{AO} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

or if $AQ = u$, and $AQ' = v$,

$$\frac{\mu'}{v} - \frac{\mu}{u} = \frac{\mu' - \mu}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Again we may obtain a similar formula connecting distances measured from O. From the law of refraction we have

$$\mu \sin ORQ = \mu' \sin ORQ'$$

and dividing both by $\sin ROQ$, we get

$$\mu OQ/RQ = \mu' OQ'/RQ',$$

or, to our order of approximation,

$$\mu OQ \,.\, AQ' = \mu' OQ' \,.\, AQ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)^*.$$

From this, if we express all lengths as distances from A we get result (2); and in terms of distances from O,

$$\mu OQ\,(OQ' - OA) = \mu' OQ'\,(OQ - OA).$$

Dividing by $OQ \,.\, OQ' \,.\, OA$ this gives

$$\frac{\mu'}{OQ} - \frac{\mu}{OQ'} = \frac{\mu' - \mu}{OA} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5);$$

or if $OQ = p$, $OQ' = q$ and $OA = r$,

$$\frac{\mu'}{p} - \frac{\mu}{q} = \frac{\mu' - \mu}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

In applying the foregoing formulae due regard must be paid to the convention of signs; and it is to be remembered that in (3) r means AO, but in (6) r means OA.

61. Principal Foci and Focal Planes.

From (2) it is clear that if the ray after refraction becomes parallel to the axis so that $AQ' = \infty$, the incident ray must cross the axis at a point F_1, such that $AF_1 = -\dfrac{\mu\, AO}{\mu' - \mu}$,

or $$AF_1 : F_1O = -\mu : \mu'.$$

* It may be observed that μ/μ' is one of the values of the cross ratio of the range $(AQOQ')$, viz. $AQ \,.\, OQ'/Q'A \,.\, QO$.

In like manner a ray which is parallel to the axis at incidence, *i.e.* a ray for which $AQ = \infty$, will after refraction cut the axis at a point F_2, such that $AF_2 = \dfrac{\mu' AO}{\mu' - \mu}$,

or $$AF_2 : F_2O = -\mu' : \mu.$$

The points F_1, F_2 are called the **first and second principal foci** of the refracting surface.

If Q, Q' are any pair of conjugate points, we have

$$\begin{aligned} F_1Q \,.\, F_2Q' &= (AQ - AF_1)(AQ' - AF_2) \\ &= \left(u + \frac{\mu r}{\mu' - \mu}\right)\left(v - \frac{\mu' r}{\mu' - \mu}\right) \\ &= \frac{uvr^2}{(\mu' - \mu)^2}\left(\frac{\mu' - \mu}{r} + \frac{\mu}{u}\right)\left(\frac{\mu' - \mu}{r} - \frac{\mu'}{v}\right) \\ &= \frac{uvr^2}{(\mu' - \mu)^2}\left(\frac{\mu'}{v}\right)\left(-\frac{\mu}{u}\right), \text{ from (3)} \\ &= \frac{-\mu\mu' r^2}{(\mu' - \mu)^2} \\ &= F_1A \,.\, F_2A. \end{aligned}$$

Planes through F_1, F_2 at right angles to the axis are called the **first and second principal focal planes** of the refracting surface.

62. Images and linear magnification.

Images are formed in the manner explained in Art. 53 and we have the following expressions for the linear magnification

$$m = \frac{P'Q'}{PQ} = \frac{OQ'}{OQ} = \frac{q}{p};$$

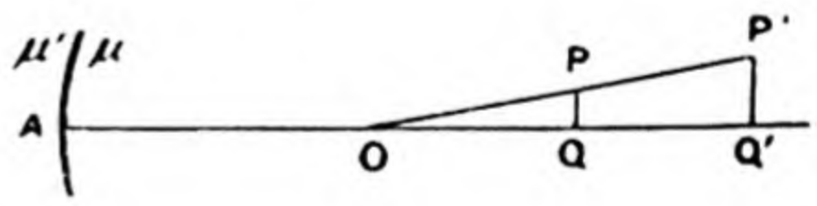


Fig. 44.

also since

$$\mu OQ \,.\, AQ' = \mu' OQ' \,.\, AQ, \quad \text{Art. 60 (4)},$$

therefore $$m = \frac{AQ'/\mu'}{AQ/\mu} = \frac{v/\mu'}{u/\mu}.$$

A distance divided by the corresponding refractive index is commonly called a *reduced distance*, so the linear magnification is the ratio of the distances of the image and object from the centre, or the ratio of their reduced distances from the vertex of the interface.

63. Longitudinal Magnification.

In the same sense as in Art. 54 we have

$$m' = \frac{\delta v}{\delta u}, \text{ where } \frac{\mu'}{v} - \frac{\mu}{u} = \frac{\mu' - \mu}{r},$$

so that

$$\frac{\mu' \delta v}{v^2} - \frac{\mu \delta u}{u^2} = 0,$$

and therefore

$$m' = \frac{\mu}{\mu'} \frac{v^2}{u^2} = \frac{\mu'}{\mu} m^2.$$

64. An incident pencil of parallel rays making small angles with the axis after refraction passes through a point on the second focal plane.

For if PO be that ray of the pencil which passes through the centre O, for this pencil we may regard PO as the axis of the interface and after refraction the pencil will pass through a point F_2' on PO such that

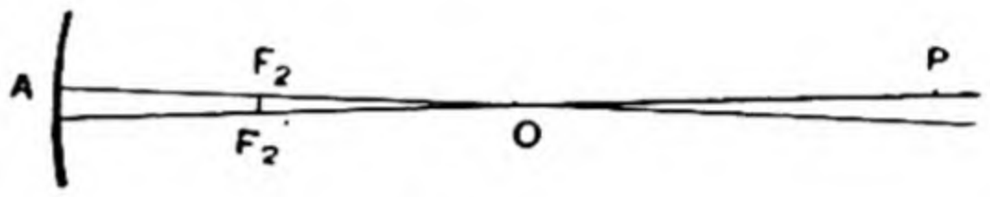


Fig. 45.

$$OF_2' = OF_2.$$

To our order of approximation this makes OF_2F_2' a right angle and F_2' a point on the second focal plane.

Similarly a pencil which before refraction passes through a point near to the axis on the first focal plane will, after refraction, become a pencil of parallel rays.

65. We observe that when $\mu' > \mu$ and the surface is concave to the incident light F_1 falls on the left and F_2 on the right of A, since

$$AF_1 = -\mu AO/(\mu' - \mu) \text{ and } AF_2 = \mu' AO/(\mu' - \mu).$$

But if $\mu' > \mu$ and the surface is convex to the incident light F_1 falls on the right and F_2 on the left of A.

The effects produced on an incident pencil of parallel rays are to produce divergence in the former case and convergence in the latter, as is clear from the figures.

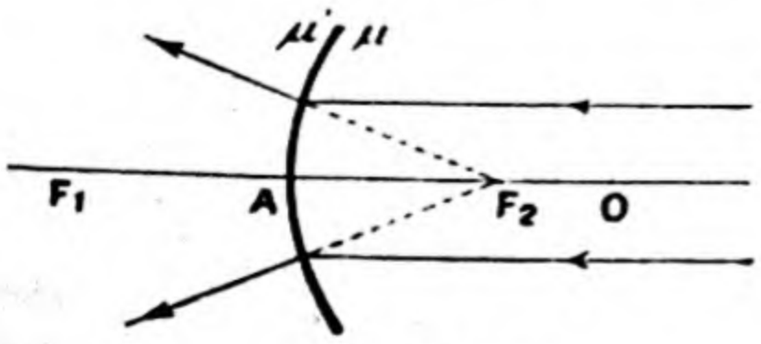


Fig. 46 (i).

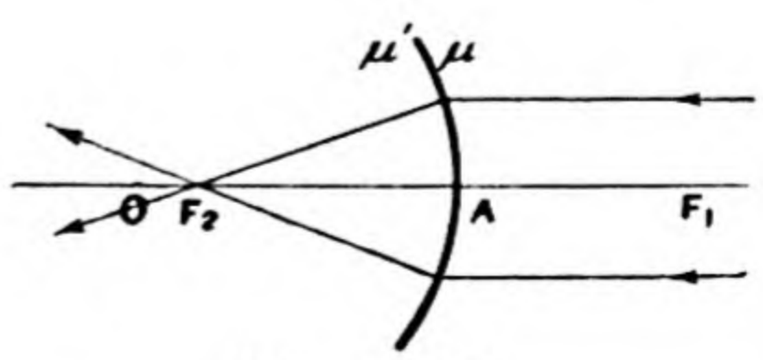


Fig. 46 (ii).

66. To construct the image of a small object.

Let PQ be a small object represented by a straight line at right angles to the axis at Q.

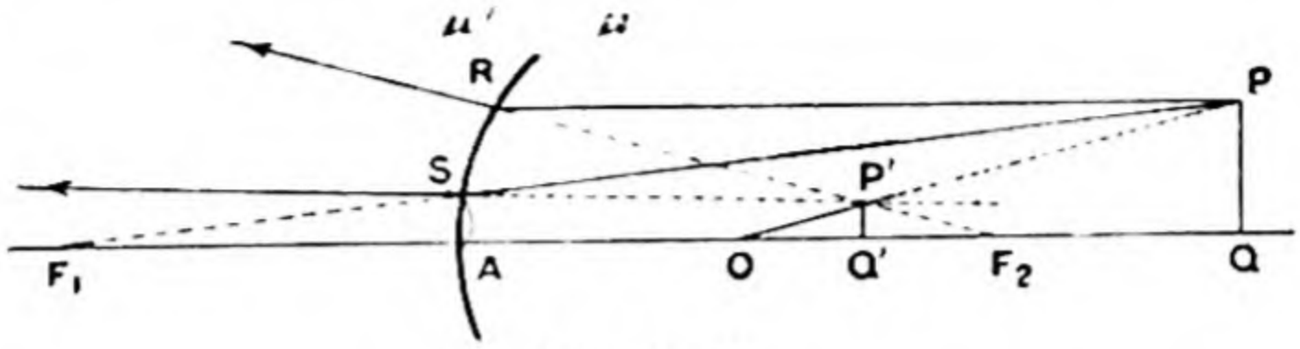


Fig. 47 (i).

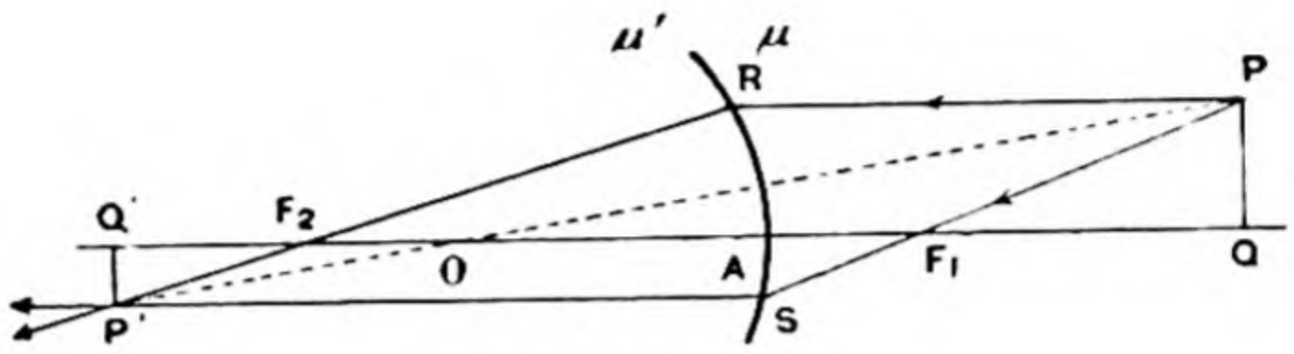


Fig. 47 (ii).

A ray PR parallel to the axis proceeds after refraction in a direction through the second principal focus F_2. A ray PS incident in direction PF_1 after refraction proceeds parallel to the axis. If the two emergent rays intersect in P' then P' is the image of P, and $P'Q'$ perpendicular to the axis is the image of PQ. There is a further check that the line PP' must pass through O the centre of the interface.

The two figures illustrate the cases of refracting surfaces which produce divergence and convergence respectively in pencils of parallel rays.

If m denote the linear magnification we have

$$m = \frac{P'Q'}{PQ} = \frac{SA}{PQ} = \frac{F_1A}{F_1Q},$$

also

$$= \frac{P'Q'}{RA} = \frac{F_2Q'}{F_2A}.$$

This means that the linear magnification varies inversely as the distance of the object or directly as the distance of the image from the corresponding principal focus.

From these formulae we get

$$F_1Q \,.\, F_2Q' = F_1A \,.\, F_2A,$$

as in Art. 61.

Since the product $F_1A \,.\, F_2A$ is a negative quantity, it follows that conjugate foci Q, Q' lie on opposite sides of the corresponding principal foci, also that if one of the points Q, Q' is moved the other will move in the same direction.

67. Helmholtz's formula for linear magnification.

If l, l' denote the linear magnitudes of a small object at Q and its image at Q'; and α, α' the angles of divergence of the same ray in the two media whose refractive indices are μ, μ', the refraction taking place at a spherical interface, then,

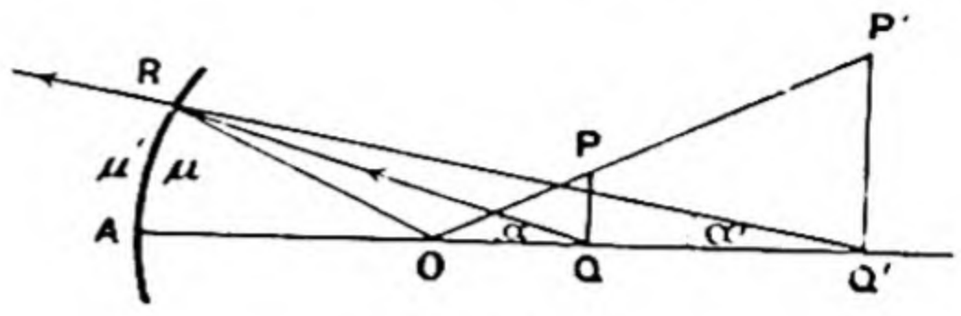


Fig. 48.

$$\mu l \sin \alpha = \mu' l' \sin \alpha'.$$

For
$$\frac{l}{l'} = \frac{OQ}{OQ'} = \frac{OQ}{OR} \cdot \frac{OR}{OQ'} = \frac{\sin ORQ}{\sin \alpha} \cdot \frac{\sin \alpha'}{\sin ORQ'}$$
$$= \frac{\mu' \sin \alpha'}{\mu \sin \alpha}.$$

Therefore
$$\mu l \sin \alpha = \mu' l' \sin \alpha',$$
or, to our order of approximation,
$$\mu l \alpha = \mu' l' \alpha'.$$

It follows that if any number of media are separated by spherical surfaces with their centres on the same axis and are traversed by rays making small angles with the axis proceeding from a small object in one of the media, then $\mu l \alpha$ is constant, when l denotes the linear dimensions of the image formed in the medium of refractive index μ, and α the divergence of that part of a ray that lies in this medium*.

EXAMPLES.

1. A man holds half-way between his eye and a convex mirror, 3 feet from his eye, two fine parallel wires so that they are seen directly and by reflection in the mirror. Prove that if their apparent distance apart as seen directly is five times what it is as seen by reflection, the radius of the mirror is 3 feet. (M. Sp.)

2. A scale divided into equal parts is placed at a distance a from a concave mirror at right angles to its axis: to an eye at a distance b beyond the scale, n divisions of the scale appear to cover $(n+1)$ divisions of its inverted image. Find the focal length in terms of n, a, b. (Coll. Exam. 1905.)

3. A man holding a finely graduated rule between his eye and a convex mirror and looking at the rule and its image by direct reflection, notices that when the former is 3 feet from his eye, one graduation on it appears as large as three on the image, and that when the rule is 4 inches nearer, one graduation appears as large as four on the image: find the radius of the mirror and the distance of its centre from his eye. (Coll. Exam. 1909.)

* H. von Helmholtz, *Handbuch der physiologischen Optik*, 1867, p. 50.

4. Two spherical mirrors of the same radius r, one concave and the other convex, are placed on a common axis ; their reflecting faces are opposite to one another and the distance between them is $2r$. A small circle of diameter d is drawn on the convex mirror about its pole as centre. Shew that the diameters of the first, second and third images of the circle are respectively $\frac{1}{3}$, $\frac{1}{11}$, $\frac{1}{41}$ of d. (M. T. i. 1911.)

5. A bright point is placed between two concentric equal concave mirrors ; shew that the distances from the centre of successive images decrease in harmonic progression.

6. An eye in front of a concave mirror whose radius is 2 feet sees an upright image of itself at a distance of 10 inches. Shew that, if the diameter of the pupil is $\frac{1}{3}$rd of an inch, the apparent angular semi-diameter of its image is $\tan^{-1}\frac{1}{40}$. (M. T. 1893.)

7. Two spherical mirrors, one concave of radius 15 inches, the other convex of radius 3 inches, face one another at a distance of 9 inches, and a small ring-shaped object is placed midway between them. Find the sizes of the images formed by successive reflections. (St John's Coll. 1905.)

8. Two concave spherical mirrors, radii R and r, are placed face to face, their distance along the principal axis being $\frac{1}{2}(R+r)$. A ray from a point on this axis at a distance $\frac{1}{2}R$ from the first mirror is incident on this mirror, making a small angle a with the axis. Shew that after $2n$ reflections its inclination will be

$$aR^n/r^n.$$

(Coll. Exam. 1902.)

9. A plane and a concave mirror are placed opposite one another on the same axis at a distance apart greater than the radius of the mirror ; a person standing with his back to the plane mirror, close to it, observes the three brightest images of a candle he holds in his hand. He moves the candle forward till it coincides with the nearest image ; prove that the other two images will coincide at the same time. He then moves the candle still further forward a distance x till it coincides with another image ; find at this instant the position of the first image, and prove that if a be the distance between the mirrors, the radius of the concave mirror is

$$\tfrac{1}{2}\{a+2x\pm\sqrt{(a^2-4ax)}\}.$$

(Coll. Exam. 1906.)

10. A thin pencil diverging from a point in air at distance u in front of a block of glass, of refractive index μ, is directly refracted into the glass. The surface of the glass is concave and of radius r. The cross sections of the refracted pencil and of the prolongation of the incident pencil by a plane at distance t behind the interface, are circles of radii ρ' and ρ. Prove that

$$\frac{\rho'-\rho}{\rho}=\frac{(\mu-1)\,t\,(u-r)}{\mu r\,(t+u)}.$$

(M. T. i. 1909.)

11. A cubical block of glass has a concentric spherical cavity and a small object in the cavity on a diameter perpendicular to a pair of opposite faces of

the cube is viewed successively through these faces. Prove that the distance apart of the images seen is

$$2\left(1-\frac{1}{\mu}\right)\left\{a+\frac{rc^2}{\mu^2r^2-(\mu-1)^2c^2}\right\},$$

where r is the radius of the sphere, $2a$ the edge of the cube, c the distance of the object from the centre, and μ the refractive index of glass; the medium inside the cavity being the same as that outside the cube.

(Coll. Exam. 1904.)

12. At each point of the axis of a spherical refracting surface an ordinate is erected to represent the distance of the point from its conjugate point: prove that the curve of its extremity is an hyperbola and determine the maximum and minimum values of the distance. (Coll. Exam. 1892.)

13. Prove the following construction for the geometrical focus q, of a small pencil emergent from a point Q, and directly refracted at the point A of a spherical surface whose centre is C, and refractive index μ: join an arbitrary point O to A and C; draw a line parallel to OQ to cut OA, OC in A' and C' respectively; produce $A'C'$ to q', so that $C'q'=A'C'/(\mu-1)$; then q is on Oq'.

(M. T. 1897.)

14. Having given two pairs of conjugate foci, for direct refraction at a spherical surface, and the vertex of the surface, find a geometrical construction for the centre of the spherical surface and the principal foci.

(Coll. Exam. 1902.)

15. Draw a curve to shew the relation between the linear magnification of an object of refraction at a spherical surface separating media of refractive indices μ, μ' and the distance u of the object from the refracting surface, for positive values of u, considering the two cases $\mu'>\mu$ and $\mu'<\mu$.

(M. T. 1907.)

16. An eye is placed close to the surface of a sphere of glass ($\mu=1{\cdot}5$) which is silvered at the back; shew that the image which the eye sees of itself is three-fifths of the natural size.

17. If r be the radius of a spherical refracting surface, l the least distance between two conjugate foci, the refractive index is $\left(\frac{r+l}{r-l}\right)^2$.

(Coll. Exam. 1888.)

18. If after refraction at a concave spherical interface of radius r between two media there be direct reflection at a concave coaxal mirror of radius $\frac{\mu r}{\mu-1}+t$ at distance t behind the first surface and then refraction back again into the first medium, shew that the total effect is the same as a single reflection at a plane mirror properly placed. (Coll. Exam. 1906.)

19. A glass tumbler is very nearly a cylinder; prove that if a pattern on the back is viewed through water ($\mu=\frac{4}{3}$), the image is double the size of the pattern. Find the position of the image and whether it is erect or inverted. Obtain the corrections to be made if the thickness of the glass is allowed for, assuming its refractive index to be $\frac{3}{2}$ and the thickness to be $1/n$th of the nternal radius.

(St John's Coll. 1911.)

20. A solid glass hemisphere has the inside of its plane face silvered except over a very small part at the centre, and light is incident on the convex surface. Shew that the images seen (by eyes in suitable positions) by refraction through the sphere, and by two refractions and a reflection occupy the same position, provided the origin of light be distant from the surface of the sphere $\frac{3}{1}\frac{3}{7}r$, where r is the radius of the sphere and $\frac{3}{2}$ is the coefficient of refraction from air to glass. (Coll. Exam. 1891.)

21. An observer, who cannot see objects at a distance less than d, views his own face in a convex mirror of radius r. Prove that the magnification cannot exceed $r/\{d+\sqrt{d^2+r^2}\}$. (M. T. 1899.)

22. A solid circular glass cylinder has a concave spherical end to which the axis of the cylinder is normal. Arranged along this axis at distances from the end which are in arithmetical progression are specks in the glass which all appear of the same size when seen through the spherical end. Prove that their actual sizes are in arithmetical progression. (M. T. 1900.)

23. A concave glass mirror bounded by two concentric spheres is silvered at the back; shew that the displacement of an image, due to the thickness t of the glass, is approximately

$$\frac{2x^2}{r^2}\frac{\mu-1}{\mu}t$$

towards the centre of curvature, where x is the distance of the image from that point and r is the radius of curvature. (M. T. 1887.)

24. A ray strikes the plane face of a glass hemisphere of refractive index μ nearly normally and is directed at incidence towards the vertex of the hemisphere; the ray is partly refracted and partly reflected at the curved surface, the two parts emerging at the curved and plane surfaces respectively; prove that their angles of divergence are in the ratio

$$\mu^2-\mu+1 : \mu-2\mu^2.$$ (M. T. 1904.)

CHAPTER V

LENSES

68. Definition. A **lens** is a portion of a transparent refractive medium bounded by two spherical surfaces. The line joining the centres of the spheres is called the **axis of the lens**.

We shall confine our attention to rays which make small angles with the axis and adopt the approximations and conventions of sign of the last chapter.

The thickness of a lens is the portion of the axis intercepted between the surfaces, and the points in which the axis cuts the lens are the **vertices**. In this chapter we shall confine our calculations to **thin lenses**, *i.e.* to lenses whose thickness is negligible compared to the radii of the spheres that bound them.

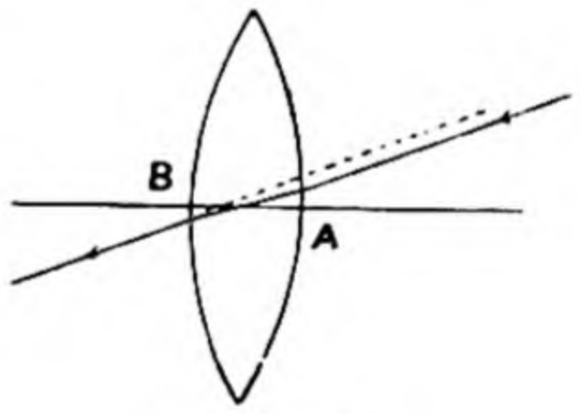


Fig. 49.

Small portions of the surfaces at the vertices are approximately parallel planes, so that a ray passing through the lens at or very near to the vertices will not undergo any deviation but will pass through the lens as through a plane sheet of glass.

69. Formulae for a thin lens.

Let μ be the refractive index and r, s the radii of the first and second surfaces on which the light falls reckoned positive when the surfaces are concave to the incident light.

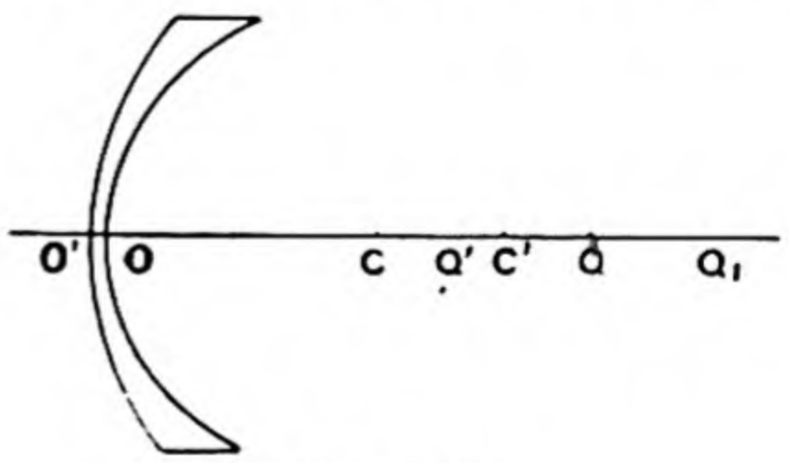


Fig. 50.

If a small object be placed at a point Q on the axis, rays from it undergo two refractions, from air into glass forming an image at Q_1 and from glass into air again forming an image at Q'. It is the final

image at Q' that concerns us, the intermediate one at Q_1 being only a connecting link between the object and the final image.

If O, O' are the vertices, for a thin lens OO' is neglected.

Let $OQ = u$, $OQ_1 = v_1$ and $OQ' = v$, all reckoned positive when drawn to meet the incident light.

For the refraction from air to glass we have

$$\frac{\mu}{v_1} - \frac{1}{u} = \frac{\mu - 1}{r},$$

and for the refraction from glass to air, measuring from O' or O (since OO' is negligible),

$$\frac{1}{v} - \frac{\mu}{v_1} = \frac{\mu - 1}{s}.$$

Hence by addition

$$\frac{1}{v} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The points Q, Q' are **conjugate foci**.

If Q' be at an infinite distance, that is if the rays from Q emerge parallel to the axis, we must put $v = \infty$ in (1); this determines a position F_1 for Q given by

$$-\frac{1}{OF_1} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Similarly if the incident rays are parallel to the axis, we must put $u = \infty$ in (1); this determines a position F_2 for Q' given by

$$\frac{1}{OF_2} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The points F_1, F_2 are called the **principal foci** of the lens. The expression $(\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right)$ is called the **power** of the lens and its reciprocal f is called the **focal length** *.

Equation (1) may now be written

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Since $\mu > 1$ the sign of the focal length depends on that of $1/r - 1/s$.

*** It must always be remembered that because of our convention about small quantities this expression for focal length refers to the central part of the lens only. A better expression will be found in Art. 152 below.**

Hence the following cases occur:—

(1) r and s both positive and $r < s$ make f positive. This is the type shewn in Fig. 50, C and C' being the centres.

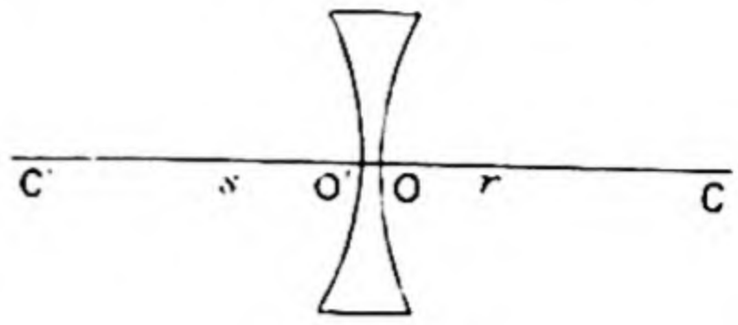


Fig. 51.

(2) r and s both negative and $r > s$ make f positive. This is a lens of the type shewn in Fig. 50, but reversed so as to be convex to the incident light.

(3) r positive and s negative make f positive. This type is shewn in Fig. 51.

(4) r and s both positive and $r > s$ make f negative. This is the type shewn in Fig. 52 (i).

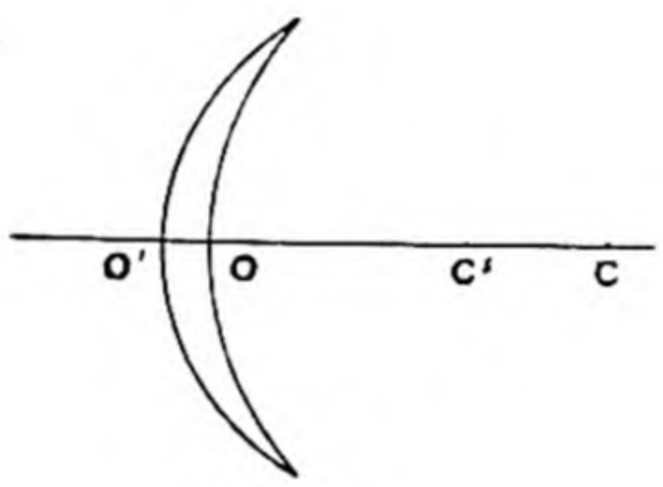


Fig. 52 (i).

(5) r negative and s positive make f negative. This is the type shewn in Fig. 52 (ii).

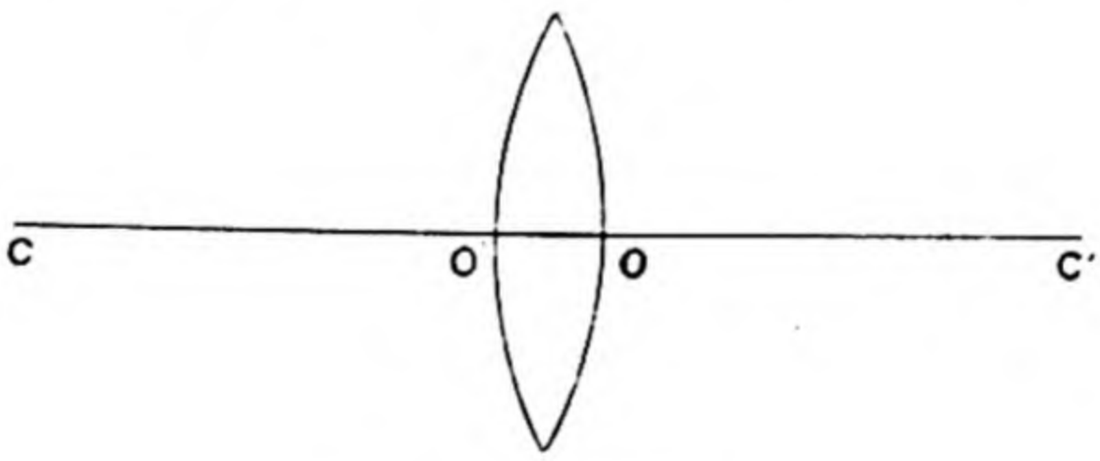


Fig. 52 (ii).

Since $OF_1 = -f$ and $OF_2 = f$, therefore when f is positive the first principal focus F_1 lies behind the lens and the second principal focus F_2 lies in front of it, meaning by the front of the lens the side turned towards the incident light. We also see that when f is positive the principal foci are *virtual* foci, for the actual paths of the rays do not pass through the foci.

The opposite holds when f is negative; and in this case the foci are real.

70. Lenses are thus divided into two main types according as their focal lengths are positive or negative. If we consider the effect of both types on a ray incident parallel to the axis we see

Virtual foci
Focal length positive
Divergent lens

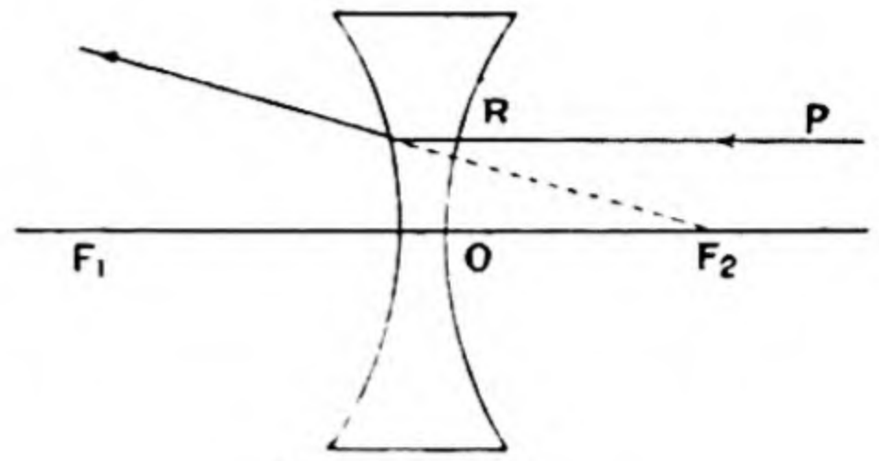


Fig. 53 (i).

Real foci
Focal length negative
Convergent lens

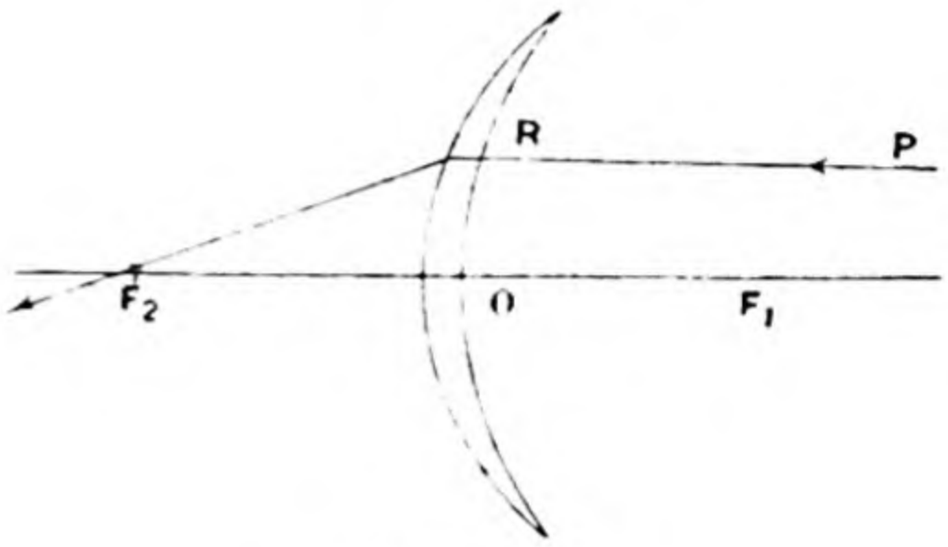


Fig. 53 (ii).

that after passing through the lens the ray diverges from or converges towards the axis according as the focal length is positive or negative. For this reason the names **divergent** and **convergent** are applied to lenses according as their focal lengths are positive or negative.

The names *concave* and *convex* are frequently used instead of divergent and convergent, but there is no apparent reason why the term concave should be applied to the lens of Fig. 50 and the term convex to that of Fig. 52 (i), and the terms divergent and convergent are preferable.

71. A lens is divergent or convergent according as its thinnest or thickest part is on its axis.

This is apparent from the figures for the different types of

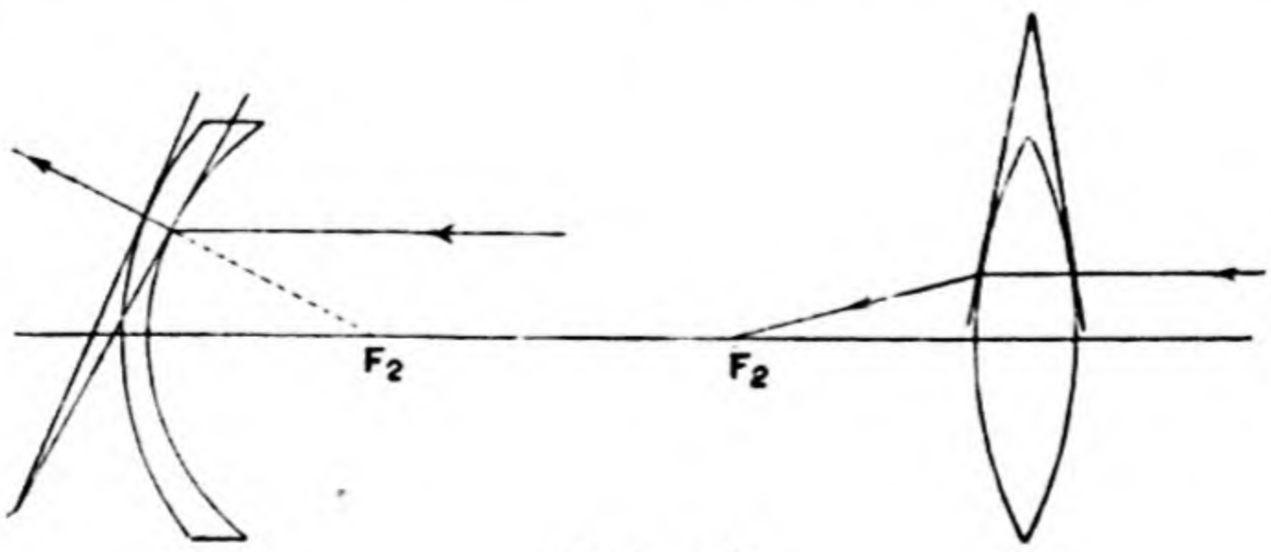


Fig. 54.

lenses according to the classification we have made. It also follows from the fact that a ray passing through a lens may be regarded as passing through a prism bounded by the tangent planes to the lens at the points of incidence and emergence, and the edge of the prism will be towards or away from the axis of the lens according as the lens has its thinnest or thickest part on the axis. But the deviation produced by a prism is always away from the edge, therefore a ray parallel to the axis is bent away from the axis of a lens that is thinnest at the axis and towards the axis of a lens that is thickest at the axis, so the former is a divergent and the latter a convergent lens.

72. Opticians' measure of power.

In practical optics a lens is of unit power if it brings parallel rays to a focus at a distance of one metre. This unit of converging power is called the **diopter**. Thus a lens is said to be of power $+1D$ if it is a convergent lens of focal length 1 metre, of power $+4D$ if it is a convergent lens of focal length 25 cms., of power $-4D$, if it is a divergent lens of focal length 25 cms. and so on. It will be observed that opticians count the focal lengths of convergent and divergent lenses as positive and negative respectively in contrast to the usage in books on theoretical optics.

73. Relation connecting distances of conjugate points from corresponding foci.

If Q, Q' are a pair of conjugate points, we have

$$F_1Q \,.\, F_2Q' = (OQ - OF_1)(OQ' - OF_2)$$

$$= (u + f)(v - f)$$

$$= uvf^2 \left(\frac{1}{f} + \frac{1}{u}\right)\left(\frac{1}{f} - \frac{1}{v}\right)$$

$$= uvf^2 \left(\frac{1}{v}\right)\left(-\frac{1}{u}\right) \text{ from Art. 69 (4).}$$

Therefore $F_1Q \,.\, F_2Q' = -f^2$.

Hence the conjugate points Q, Q' are on opposite sides of corresponding foci and if one moves the other moves in the same direction.

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74. Construction of the image of a small object. Linear magnification.

Virtual foci
Divergent lens
f positive

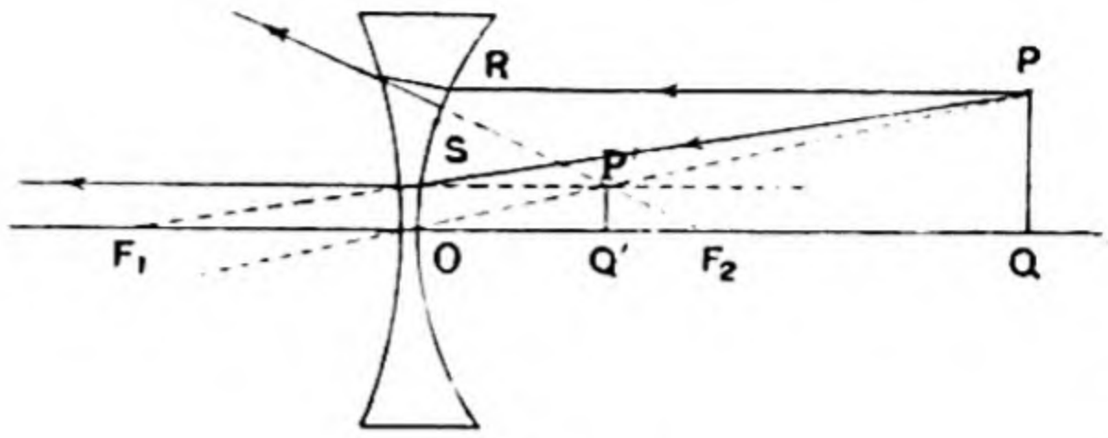

Fig. 55 (i).

Real foci
Convergent lens
f negative

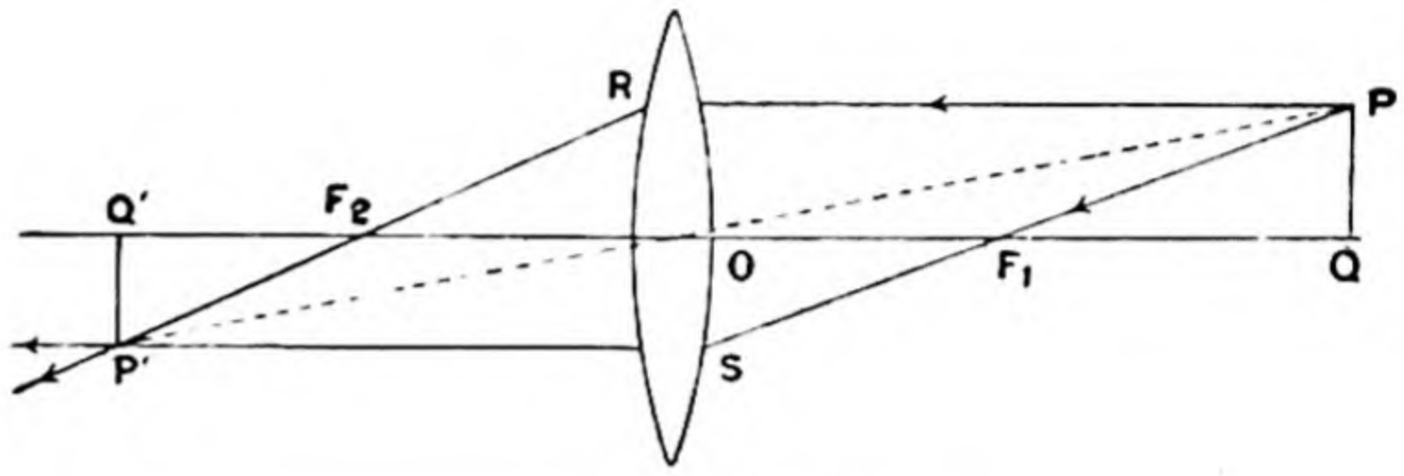

Fig. 55 (ii).

Let PQ be a small object represented by a straight line at right angles to the axis at Q. A ray PR incident parallel to the axis, after refraction proceeds in the direction F_2R or RF_2, and a ray PS incident in the direction PF_1 after refraction proceeds parallel to the axis. If these two emergent rays intersect in P', then P' is the image of P and $P'Q'$ at right angles to the axis is the image of PQ. We have the further verification that since a ray passes through the vertex of a lens without deviation when the lens considered is thin, therefore P, P' are on a line through O the vertex of the lens.

For the linear magnification we have

$$m = \frac{P'Q'}{PQ} = \frac{SO}{PQ} = \frac{F_1O}{F_1Q},$$

also

$$= \frac{P'Q'}{RO} = \frac{F_2Q'}{F_2O}.$$

That is the linear magnification varies inversely as the distance of the object or directly as the distance of the image from the corresponding focus.

From the above we deduce at once Newton's formula

$$F_1Q \,.\, F_2Q' = F_1O \,.\, F_2O = -f^2,$$

as given in the last article.

Another expression for the linear magnification is

$$m = \frac{P'Q'}{PQ} = \frac{OQ'}{OQ} = \frac{v}{u}.$$

Planes at right angles to the axis through F_1, F_2 are called **principal focal planes.**

75. By placing the object PQ at F_1, the foregoing construction makes the image infinite and at an infinite distance. Hence if rays, at incidence on a divergent lens, are converging towards a point P in the first focal plane they emerge as a pencil of parallel rays; and if rays, at incidence on a convergent lens, are diverging from a point P in the first focal plane they emerge as a pencil of parallel rays; the direction of the emergent pencil being given in both cases by the ray OP which undergoes no deviation.

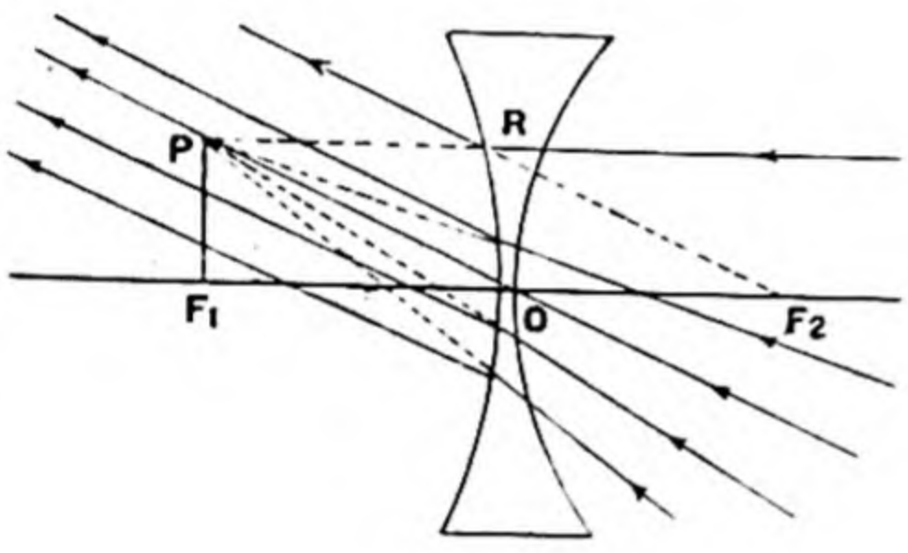

Fig. 56 (i).

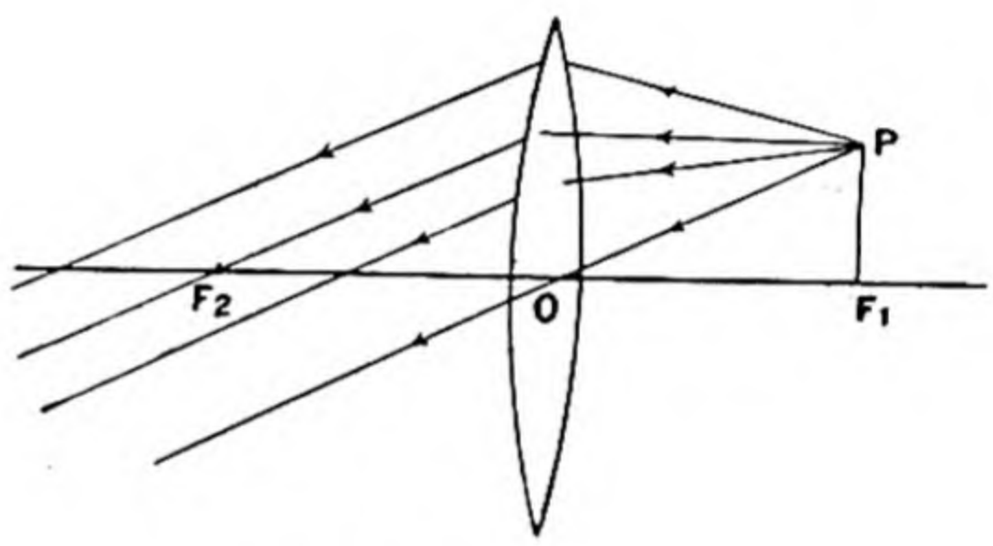

Fig. 56 (ii).

76. We can now investigate the position and magnitude of the image of a small object on the axis of a thin lens for different positions of the object.

For a **divergent lens** referring to Fig. 55 (i) we see that, for a real object, since F_1Q must be greater than F_1O, therefore F_2Q' must be less than F_2O and object and image being on opposite sides of F_1 and F_2 respectively, it follows that the image will always be virtual and erect lying between O and F_2. When the object is at the lens the image coincides with it and as the object moves away through F_2 to an infinite distance the image moves from O to F_2, gradually decreasing in size.

For a **convergent lens** we see that for a real object beyond the first focus F_1, there is a real image behind the second focus F_2. As the object moves up to F_1, the image increases in size and recedes from F_2, becoming infinite in size and infinitely

distant when the object reaches F_1. As soon as the object passes F_1, since F_1Q is then less than F_1O, therefore F_2Q' must be greater than F_2O which means that the image will be in front of the lens, erect and virtual. When F_1Q is very small F_2Q' must be very great, so that as Q passes F_1 in its approach to the lens the virtual image is at first very large and very distant. As Q approaches O, F_1Q increases and therefore F_2Q' decreases; so the image decreases and moves nearer and this continues until the object reaches O where the image coincides with it.

This investigation may also be made by observing that in Figs. 55, for different positions of the same object PQ, the emergent ray RF_2 is fixed in position, and the point P' of the image is got as the intersection of PO and RF_2.

77. Angular magnification. When a lens is used for viewing an object, the ratio of the angles subtended at the eye by the image and the object is called the angular magnification.

78. Longitudinal magnification.

We have $$m' = \frac{\delta v}{\delta u}, \quad \text{where } \frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

so that $$\frac{\delta v}{v^2} - \frac{\delta u}{u^2} = 0,$$

and therefore $$m' = \frac{v^2}{u^2} = m^2.$$

EXAMPLES.

1. Light converging towards a point A is intercepted before reaching A by a lens at O, and after passing through the lens converges to B. If $OA = 6$ in., and $OB = 12$ in., find the nature and the focal length of the lens.
(St John's Coll. 1907.)

2. The two surfaces of a thin double convex lens for which $\mu = 1{\cdot}5$ are of the same radius and rays diverging from a bright point at 6 inches from it appear, after passing through the lens, to diverge from a point at 15 inches from it. Find the radius of each surface. (Coll. Exam. 1909.)

3. An object and its image formed by a thin lens are 32 inches apart; the image is inverted and its linear dimensions are 3 times those of the object. Find the position and the focal length of the lens. (St John's Coll. 1908.)

4. A thin convergent lens is found to give an inverted image of an object the same size as the object when this is 20 cms. from the lens. An image formed by reflection from the second surface of the lens is found to be

coincident with the object when the object is 6 cms. from the lens. Calculate the radius of curvature of the second surface of the lens, and, assuming $\mu = 1{\cdot}5$, find the radius of curvature of the first surface. (M. T. i. 1908.)

5. In a magic lantern the image of a slide is thrown upon a sheet by means of a thin convex lens. Shew that the adjustment for focussing is always possible provided that the distance from the slide to the sheet is not less than four times the focal length of the lens and that the lens may move in its tube to a distance of twice its focal length from the slide.

(Coll. Exam. 1907.)

6. A thin convergent lens casts a real image of a candle-flame on a screen; the lens is displaced through a distance c (the candle and screen remaining fixed) until the image on the screen is again sharp. Prove that the focal length of the lens is

$$\tfrac{1}{4}(d - c^2/d),$$

where d is the distance from the candle to the screen.

(St John's Coll. 1909.)

7. A person handling a lens notices that by reflection at the nearer surface he sees objects which are situated about 6 ft. behind him, upright and apparently diminished to $\frac{1}{20}$ of their actual linear dimensions; looking through the lens, he sees objects that are about 6 ft. beyond it, inverted and diminished to $\frac{1}{10}$ of their actual linear dimensions. Find the radii of curvature of the faces of the lens, assuming $\mu = \frac{3}{2}$ for glass.

(St John's Coll. 1904.)

8. It is desired to concentrate the sun's rays at a point on the axis of a convergent thin lens, at a distance c from the lens on the same side as the sun. Prove that this can be done by the aid of a plane mirror placed at a certain distance behind the lens, provided that f does not lie between c and $2c$.

(M. T. i. 1910.)

9. A person holds a convergent lens of focal length six inches at a distance twelve inches from his eye to examine an object four inches beyond the lens. Account for the magnifying power of the lens, and shew that its numerical measure is two.

(St John's Coll. 1908.)

10. If rays are passing through a lens and at incidence are converging towards a point P behind the lens near its axis, investigate the position of the image of P corresponding to all positions from contact with the lens to an infinite distance, when the lens is (1) divergent, (2) convergent.

11. A photographic camera has a thin lens of 15 cm. focal length and of 3 cm. diameter. The camera is adjusted so that an object on the axis of the lens 240 cm. from the lens is in perfect focus on the plate. When an object point P lies on the axis at a distance greater or less than 240 cm. from the lens, the cone of rays emerging from the lens is cut by the plate in a circle. Shew that the diameter (c cm.) of this circle and the distance (p cm.) of P from the lens are connected by the relation

$$5cp = |\,240 - p\,|$$

where $|\,x\,|$ denotes the magnitude of x without regard to sign.

Find the two values of p corresponding to $c = 0{\cdot}01$. (M. T. i. 1914.)

12. A plane mirror is placed behind a convergent lens; prove that if a needle is placed at the first focal point of the lens, its image, as seen by rays which have traversed the lens twice, will coincide with the needle; and that the image is equal to the object, but is inverted. (M. T. i. 1913.)

13. Shew that, if a convergent lens be placed at a distance greater than its focal length from a convex mirror, there are two positions of a source of light for each of which the image, formed after the light has passed twice through the lens, is coincident with the object.

Shew that for one position the image is inverted and for the other it is erect. (M. T. i. 1908.)

14. A thin convex lens, focal length 2 ft., is placed in front of, and parallel to, a plane mirror, so that a real image of the lens is formed by reflection at the mirror and refraction through the lens. Shew that the least distance of the image from the mirror is $(3+2\sqrt{2})$ feet. (M. T. i. 1912.)

15. A convex lens and a concave mirror each of focal length 3 inches are at a distance of 8 inches on the same axis. Find the linear magnification of the image of the lens after reflection and refraction and shew that it is numerically the same as that of the image of the lens after reflection alone, when placed at a distance of $2\frac{2}{3}$ inches or $3\frac{1}{3}$ inches from the mirror.

Which of the images are inverted and which erect? (Coll. Exam. 1913.)

16. The numerical value of the focal length of a thin convergent lens is f, and the object is at a distance u in front of the lens; also a plane mirror is placed behind the lens perpendicular to the axis at a distance f from the lens. The image of the object after reflection at the mirror and a second refraction through the lens is at a distance v in front of the lens. Prove that

$$v+u=2f.$$ (Trinity Coll. 1904.)

17. A small object is placed at the focus of a thin convergent lens of focal length f. A concave mirror of focal length f' is placed behind the lens with its focus coinciding with the second focus of the lens. An eye on the axis at distance d from the object views it by rays passing through the lens twice with an intermediate reflection. Draw the paths of the rays and prove that the angular magnification is $f'd/f^2$. (M. T. ii. 1910.)

18. A concave spherical mirror of focal length f is placed on the axis of a thin convergent lens of focal length f', with the focus of the mirror at the centre of the lens. Prove that the combination behaves like a spherical mirror of focal length $ff'^2/(f'^2-f^2)$, and find the position of its centre. (Coll. Exam. 1911.)

19. A concave mirror A and convex lens C have the same focal length f and a common axis $AC=f$, Q is an origin of light on AC produced such that $CQ=2f$; shew that the image of Q after refraction through the lens, reflection at the mirror, and refraction back through the lens, coincides with A. (Coll. Exam. 1887.)

20. Light from a bright object A passes through a converging lens B, is reflected at a convex spherical mirror C, and returns through the lens. Prove that, if the position of C be so chosen that the final image coincides with A, but is inverted, the centre of the sphere will be at the position of the image formed by B when the mirror is removed. Under what circumstances would the final image at A have been erect?

(St John's Coll. 1908.)

21. An observer holds in front of himself a thin double convex lens, which has the radii of curvature (r) of its faces numerically equal. He sees two images of himself, one erect and the other inverted. Explain how these images are formed and, taking the index of the glass to be $\frac{3}{2}$, prove that he will see two erect images if the distance of the lens is less than $\frac{1}{4}r$.

(Coll. Exam. 1904.)

22. A plane mirror has a plate of glass, protecting the quicksilver, of thickness t and index of refraction μ. A small picture faces the mirror at distance u from the first surface of the glass. The light from the picture, after nearly direct incidence on and reflection from the mirror, is received by a thin lens also at distance u from the first surface, which produces a real image of the picture magnified n times. Prove that, neglecting any obliquity of incidence, the similar image produced by light reflected at the first surface is behind this image at a distance

$$2n^2t(\mu u + t)/\mu(\mu u - nt).$$

(M. T. 1901.)

23. A pencil of light diverging from a luminous point is refracted at one surface of a thin double convex lens, reflected at the other, and refracted out again. Shew that the final image will be inverted, except when the distance of the luminous point from the lens is $< rr'/2(\mu r + \mu - 1 . r')$ where r and r' are the radii of the two surfaces of the lens.

(Coll. Exam.)

24. A thin concave lens of focal length f and a concave spherical mirror of focal length F are placed on the same axis at a distance a apart. An eye on the axis sees its own image at a distance d, magnified m times; prove that

$$m - \frac{1}{m} = d\left(1 + \frac{a}{f}\right)\left(\frac{a}{Ff} + \frac{1}{F} - \frac{2}{f}\right).$$

(St John's Coll. 1905.)

25. Shew that the area of the image of a small triangle close to the axis of a lens, and in a plane through it, is equal to the area of the original triangle $\times$ the cube of the linear magnification.

(Coll. Exam. 1906.)

Oh. Only Headache and nothing else
so don't bother about Physics let it go to
hell

CHAPTER VI

COMBINATIONS OF TWO OR MORE THIN LENSES

79. Two thin lenses.

Let there be two thin lenses of focal lengths f, f' at points O, O' on the same axis at a distance a apart.

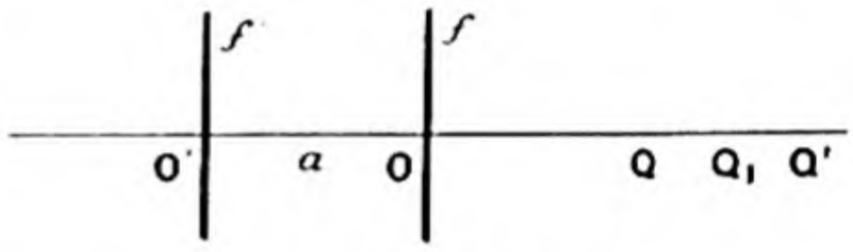

Fig. 57.

Let Q be a small object on the axis, Q_1 its image formed by the first lens and Q' the image of Q_1 formed by the second lens. Then Q' is the image of Q formed by the combination and we want to determine the position of Q' when that of Q is given.

Let $OQ = u$, $OQ_1 = v_1$, $OQ' = v$, then by Art. 69

we have $$\frac{1}{v_1} - \frac{1}{u} = \frac{1}{f} = k, \text{ say, } \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and $$\frac{1}{v} - \frac{1}{v_1 + a} = \frac{1}{f'} = k', \text{say, } \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where k, k' denote the powers of the lenses.

From (1) we get $v_1 = u/(ku + 1)$, and substituting in (2) gives

$$\frac{1}{v} = k' + \frac{1}{a + u/(ku+1)} = k' + \frac{ku+1}{u(ak+1)+a},$$

therefore $$v = \frac{u(ak+1)+a}{u(akk' + k + k') + ak' + 1} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

We may write this result

$$v = \frac{Cu + D}{Au + B} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)^*,$$

* It is worthy of observation that if we assume that to each point on the axis there is one and only one conjugate point the relation which connects their distances from fixed origins on the axis must be of the form (4), because (4) may be written

$$Auv + Bv - Cu - D = 0,$$

and this is the most general form of equation which gives one and only one value of v in terms of u and vice versa.

where
$$\left.\begin{aligned} A &= akk' + k + k', \quad B = ak' + 1, \\ C &= ak + 1, \qquad\quad D = a; \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (5),$$
and A, B, C, D are not independent because by multiplication
$$BC - AD = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Result (4) connects the distances of the object and final image from the first and second lens both measured towards the incident light.

The focal lengths f, f' are taken algebraically so that if either lens be convergent care must be taken to give its focal length a negative value in making numerical applications of the formulae.

80. Principal foci and focal planes.

The combination has **principal foci** F_1, F_2 determined as the limiting position of Q when Q' is at infinity and the limiting position of Q' when Q is at infinity.

To make v infinite we must have $Au + B = 0$, hence F_1 is determined by
$$OF_1 = -\frac{B}{A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, if we divide numerator and denominator of the right-hand side of (4) by u and then make u infinite we get $v = C/A$ so that F_2 is determined by
$$O'F_2 = \frac{C}{A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The principal foci possess the usual properties, namely, rays directed so as to pass through F_1 at incidence on the first lens will emerge from the second lens parallel to the axis; and rays incident on the first lens parallel to the axis will emerge from the second lens in directions passing through F_2.

The planes passing through the principal foci at right angles to the axis are called **principal focal planes.**

If Q, Q' are any pair of conjugate foci, we have
$$\begin{aligned} F_1Q \,.\, F_2Q' &= (OQ - OF_1)(O'Q' - O'F_2) \\ &= \left(u + \frac{B}{A}\right)\left(v - \frac{C}{A}\right) \\ &= \frac{Au + B}{A}\left(\frac{Cu + D}{Au + B} - \frac{C}{A}\right), \text{ from Art. 79 (4)} \\ &= \frac{AD - BC}{A^2} \text{ and therefore from Art. 79 (6) we get} \end{aligned}$$
$$F_1Q \,.\, F_2Q' = -\frac{1}{A^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Hence conjugate foci are on opposite sides of the corresponding principal foci, and if one is moved the other moves in the same direction.

81. Linear magnification produced by a combination of two thin lenses.

If m denote the linear magnification, it is clearly the product of the two linear magnifications due to refraction through the two separate lenses.

Therefore, from Art. 74,

$$\frac{1}{m} = \frac{u}{v_1} \cdot \frac{v_1 + a}{v}$$

and from Art. 79 (2),

$$\frac{v_1 + a}{v} = 1 + k' (v_1 + a),$$

so that

$$\frac{1}{m} = \frac{u}{v_1} \{1 + k' (v_1 + a)\}$$

$$= \frac{u}{v_1} (1 + k'a) + uk'.$$

And from Art. 79 (1) $\quad \frac{u}{v_1} = 1 + ku\,;$

hence $\frac{1}{m} = (1 + ku)(1 + k'a) + uk'$

$= u (akk' + k + k') + ak' + 1\,;$ or from Art. 79 (5)

$= Au + B$ .....(1).

This gives the magnification in terms of the distance of the object from the first lens; we may also express it in terms of v the distance of the image from the second lens. For from Art. 79 (4) and (6),

$$Auv + Bv - Cu - D = 0 \quad \text{and} \quad BC - AD = 1,$$

therefore, eliminating D,

$$A^2uv + ABv - ACu - BC + 1 = 0,$$

or $\quad (Au + B)(C - Av) = 1$ .....(2).

Hence we have $\quad m = C - Av$.....(3).

82. Unit points and planes.

We might use equations (1) and (3) of the last article to determine the position of a pair of conjugate points for which the linear magnification m had any assigned value. There is a special

pair of conjugate points so determined which are of great importance, namely those for which the linear magnification is unity, or $m = 1$. These points are called the **unit points** or points of unit magnification. We shall denote the unit points by H, H'. Thus a small object placed at H will have an image equal to itself and similarly placed at H'. Planes at right angles to the axis through the unit points are called the **unit planes***.

To find the points H, H' we have that, when $m = 1$, Q coincides with H and Q' with H'. Hence by putting $m = 1$ in (1) and (3) of the last article

we get $$OH = u = \frac{1 - B}{A}$$

and $$O'H' = v = \frac{C - 1}{A} \Bigg\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

83. Focal length of a combination of two thin lenses.

Combining equations (1) and (2) of Art. 80 with (1) of the last article, we get

$$F_1H = OH - OF_1 = \frac{1}{A},$$

and $$F_2H' = O'H' - O'F_2 = -\frac{1}{A}.$$

The distance F_1H is called the **focal length** and its reciprocal A is called the **power** of the combination.

The focal length is positive if F_1H is in the direction opposite to that of the incident light, and negative if in the same direction.

If F denote the focal length we have

$$\frac{1}{F} = A = k + k' + akk', \text{ by Art. 79 (5)},$$

therefore $$\frac{1}{F} = \frac{1}{f} + \frac{1}{f'} + \frac{a}{ff'}.$$

As a special case, if two thin lenses are in contact the power of the combination is the sum of the powers of the separate lenses.

* The use of these planes was introduced by C. F. Gauss : *Dioptrische Untersuchungen*, Goettingen, 1841, or *Werke*, B. v. p. 243. He called the points of unit magnification *principal points*, but there are obvious reasons why *unit points* is a more suitable name.

84. Formula in terms of distances from unit points.

Let $HQ = p$, and $H'Q' = q$, where Q, Q' are any pair of conjugate points.

Since $$F_1Q \,.\, F_2Q' = -1/A^2,$$

therefore $$(F_1H + p)(F_2H' + q) = -1/A^2,$$

or $$\left(\frac{1}{A} + p\right)\left(-\frac{1}{A} + q\right) = -\frac{1}{A^2}.$$

Whence we get $$\frac{1}{q} - \frac{1}{p} = A,$$

or $$\frac{1}{q} - \frac{1}{p} = \frac{1}{F}.$$

85. Divergent and convergent combinations.

The principal foci and the unit points are sometimes called the **cardinal points** of a lens or combination of lenses, because when their positions are known it is possible to determine the point conjugate to any given point or the emergent ray corresponding to any incident ray.

We observe in the first place that if an incident ray strikes the first unit plane at a given distance from the axis, the fundamental property of the unit planes, viz. equality of object and image, requires that the emergent ray shall cut the second unit plane at the same distance from the axis on the same side of the axis.

F positive

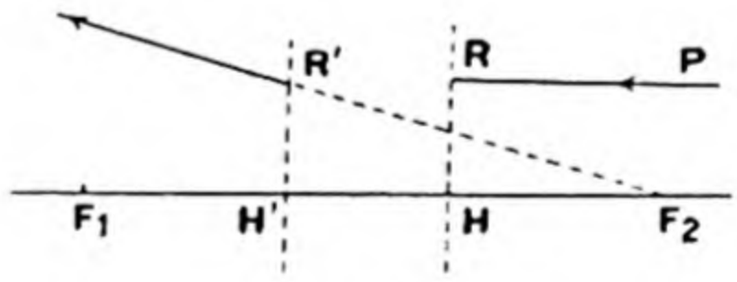

Fig. 58 (i).

F negative

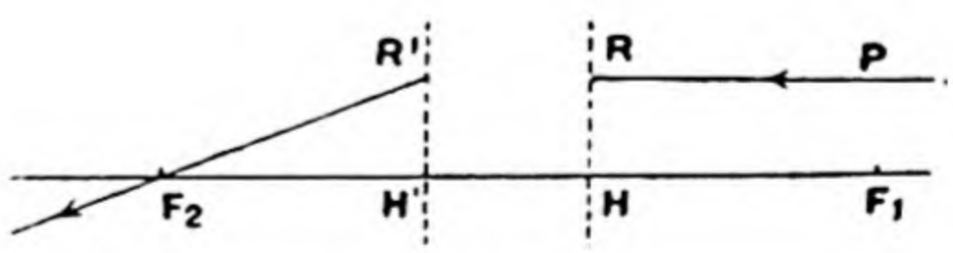

Fig. 58 (ii).

Now let a ray PR parallel to the axis cut the first unit plane in R, then the emergent ray must pass through F_2 and cut the second unit plane at R', such that $H'R' = HR$. In the first figure the focal length is positive, F_1 being to the left of H (with the usual convention about signs) and F_2 to the right of H' and the

emergent ray *diverges* from the axis as it leaves F_2. In the second figure the focal length is negative, F_1 being to the right of H and F_2 to the left of H' and the emergent ray *converges* towards the axis as it approaches F_2. But whether the rays that emerge from the second *lens* are diverging or converging depends on the position of F_2 with regard to the second lens, thus in the second figure the emergent pencil will be convergent or divergent according as the second lens is to the right or left of F_2. Hence we cannot conclude as in the case of a single lens that the ultimate divergence or convergence of rays incident parallel to the axis depends only on the sign of the focal length; what it does depend on is whether the foci are virtual or real, and it is possible for a combination of two lenses to have one focus real and the other virtual, so that parallel rays are made to diverge or converge according as which lens of the pair is turned to meet the incident light. This is illustrated by an example in Art. 91 below, wherein it is seen that F_1 is a real focus and F_2 a virtual focus, and that rays parallel to the axis are made to diverge or converge according as they are incident first on the lens at O or first on that at O'.

86. To construct the image of a small object on the axis.

F positive

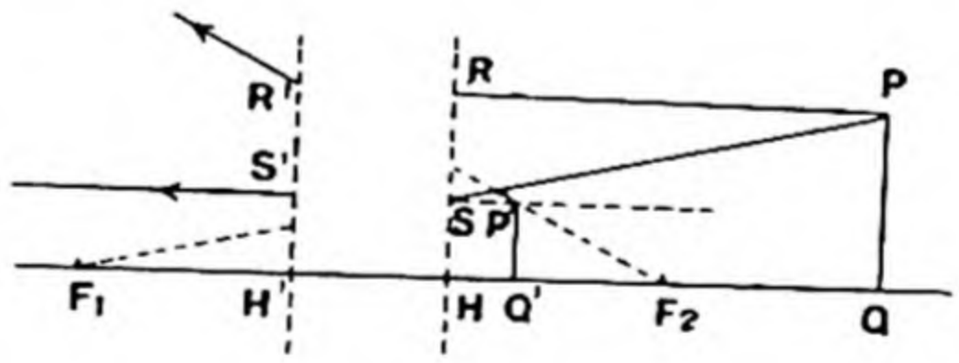

Fig. 59 (i).

F negative

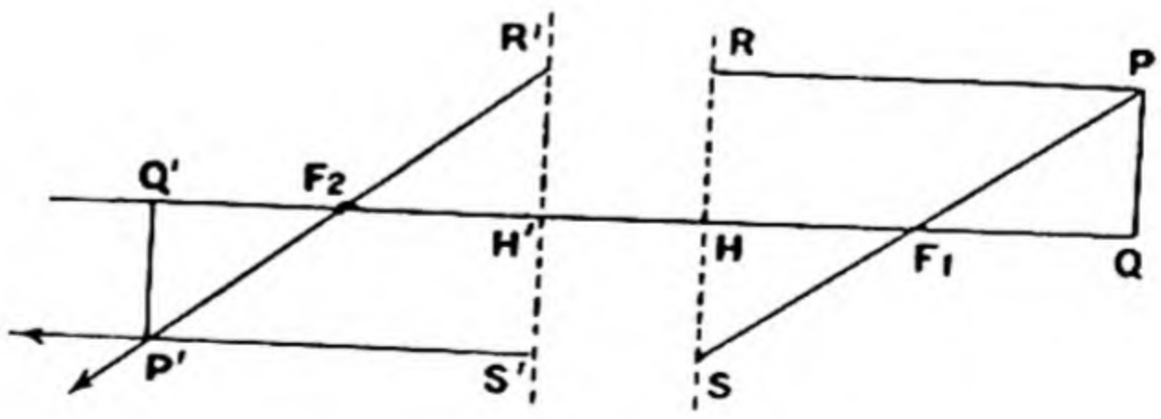

Fig. 59 (ii).

If PQ denotes a small object at right angles to the axis at Q and we draw rays PR parallel to the axis and PS directed to F_1, meeting the first unit plane in R, S respectively, the emergent rays pass through points R', S' on the second unit plane such that $H'R' = HR$ and $H'S' = HS$, and the former passes through F_2

and the latter is parallel to the axis. These two emergent rays intersect in P' the image of P, and $P'Q'$ perpendicular to the axis is the image of PQ.

From the figures we may deduce the expressions for linear magnification, for we have

$$m = \frac{P'Q'}{PQ} = \frac{SH}{PQ} = \frac{F_1H}{F_1Q} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

also
$$= \frac{P'Q'}{R'H'} = \frac{F_2Q'}{F_2H'} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

These expressions are really the same as those of Art. 81, for

$$F_1H = \frac{1}{A} \quad \text{and} \quad F_1Q = OQ - OF_1 = u + \frac{B}{A},$$

therefore
$$m = \frac{F_1H}{F_1Q} = \frac{1}{Au + B}.$$

Also $F_2H' = -\frac{1}{A}$ and $F_2Q' = O'Q' - O'F_2 = v - \frac{C}{A}$,

therefore
$$m = \frac{F_2Q'}{F_2H'} = C - Av.$$

From the foregoing expressions for m we deduce

$$F_1Q \,.\, F_2Q' = F_1H \,.\, F_2H' = -F^2 \quad \ldots\ldots\ldots\ldots\ldots(3),$$

as in Art. 80.

87. Pencils converging to or diverging from a point on the first focal plane emerge as parallel rays.

It is clear that if in the last two figures we move PQ to F_1 so that the incident rays may be regarded as, in the first figure converging towards, and in the second figure diverging from, a point P on the first focal plane the emergent ray F_2R' is unaltered, but the points S, S' are infinitely distant from the axis so that the image $P'Q'$ becomes infinite and at an infinite distance. Hence the rays incident through P must all emerge parallel to F_2R'.

That the image is infinite when Q is at F_1 also follows from (1) of the last article, and that it is at an infinite distance follows from (3).

88. Having given an incident ray, to construct the corresponding emergent ray.

If the ray meets the first unit plane in R, the emergent ray meets the second unit plane in R', where $H'R' = HR$.

Another point P' on the emergent ray can be found by constructing the image of any point P on the incident ray as in Art. 86, and then $R'P'$ is the ray required.

Otherwise we may make use of the properties of the focal planes proved in the last article.

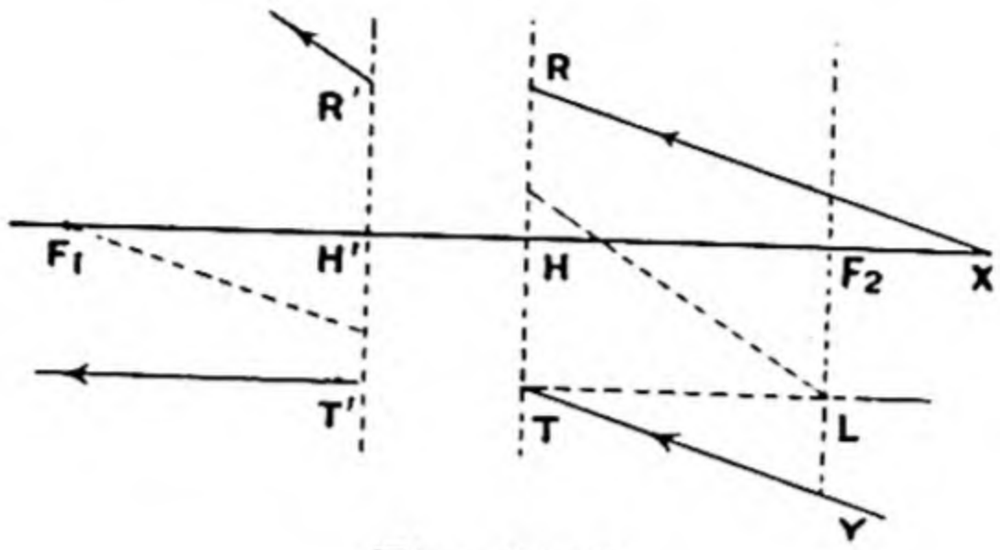

Fig. 60 (i).

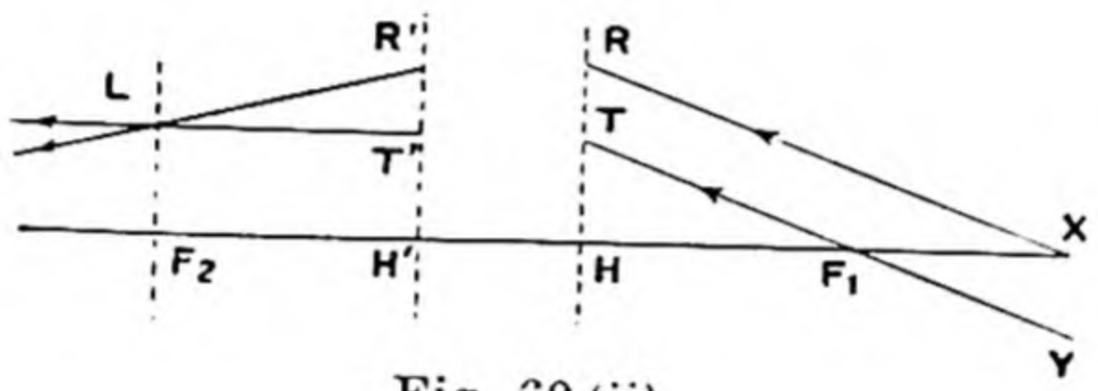

Fig. 60 (ii).

If XR is the incident ray meeting the first unit plane in R, we take a parallel ray YTF_1 or YF_1T passing through the first principal focus and meeting the first unit plane in T. This ray emerges as $T'L$ parallel to the axis, cutting the second focal plane at L. But since the two incident rays are parallel, on emerging they must both pass through the same point L on the second focal plane. Hence if $H'R' = HR$ then LR' or $R'L$ is the direction of the emergent ray.

89. Equivalent lens.

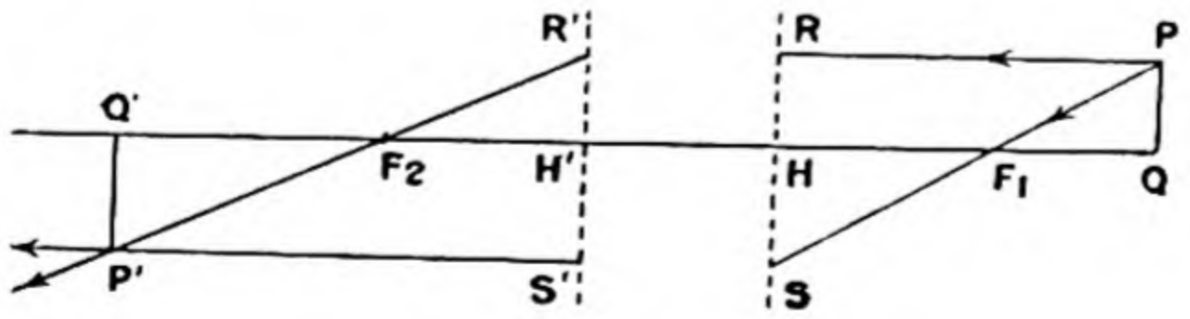

Fig. 61 (i).

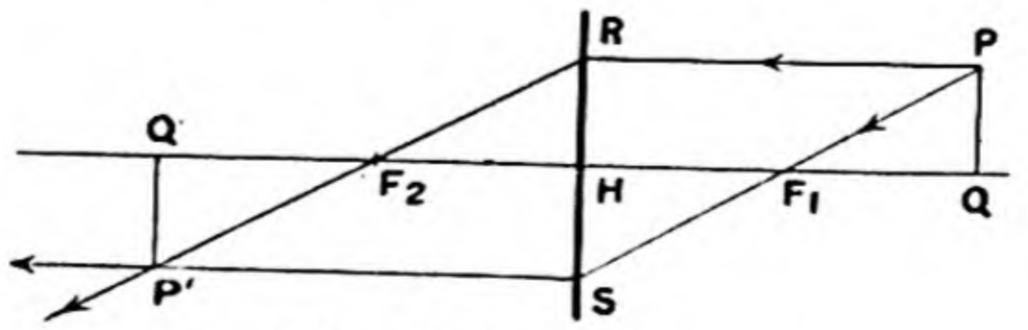

Fig. 61 (ii).

A comparison of the two accompanying diagrams which shew the construction of the image of a small object (1) formed by a combination whose cardinal points are given, (2) formed by a thin lens of the same focal length as the combination placed at the first unit point of the combination, makes it clear that this thin lens produces images of the same size as does the combination but displaced a distance $H'H$ towards the incident light. For the only difference between the two diagrams is the omission in the second of the blank strip between the unit planes in the first. A thin lens of the same focal length as a given combination and placed at its first unit point is called the **equivalent lens** of the combination. It is only *equivalent* however in the sense indicated; namely, that it produces the same linear magnification as does the combination for small objects on its axis.

90. **Numerical Examples.** For the purposes of working numerical examples it is neither necessary nor desirable to quote the various formulae that have been established in this chapter, but it is important to remember the main facts that they embody; namely, the existence of principal foci and unit points and planes and their chief properties. We shall illustrate this fact by working an example making use of no formulae save the formulae for refraction and magnification produced by a single thin lens.

91. **Example.** *An eye-piece is formed of two thin convergent lenses of focal lengths* 4 *cms. and* 10 *cms. at a distance* 7 *cms. apart. Calculate the focal length of the eye-piece and draw a diagram to indicate the position with respect to the lenses of the focal and unit points of the eye-piece.*

Draw in the diagram the course of a ray which strikes the first lens at right angles. (M. T. i. 1913.)

Referring to Fig. 57 we have $f=-4$, $f'=-10$ and $a=7$. Therefore formulae (1) and (2) for refraction through the lenses become

$$\frac{1}{v_1}-\frac{1}{u}=-\frac{1}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and

$$\frac{1}{v}-\frac{1}{v_1+7}=-\frac{1}{10} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

To find F_1, put $v=\infty$, therefore from (2) $v_1+7=10$ or $v_1=3$. Thus from (1

$$\frac{1}{3}-\frac{1}{u}=-\frac{1}{4},$$

therefore $$OF_1=u=\tfrac{12}{7}=1\tfrac{5}{7} \text{ cms.}$$

Again to find F_2, put $u=\infty$, therefore from (1) $v_1=-4$.

Then from (2)

$$\frac{1}{v}-\frac{1}{3}=-\frac{1}{10},$$

therefore $$O'F_2=v=\tfrac{30}{7}=4\tfrac{2}{7} \text{ cms.}$$

Again if m denote linear magnification,

$$\frac{1}{m} = \frac{v_1+7}{v} \cdot \frac{u}{v_1}, \text{ and from (2)}$$

$$= \left(1 - \frac{v_1+7}{10}\right) \frac{u}{v_1} = \frac{(3-v_1)\,u}{10v_1}$$

$$= \frac{3u}{10v_1} - \frac{u}{10}, \text{ and from (1)}$$

$$= \frac{3}{10}\left(1 - \frac{u}{4}\right) - \frac{u}{10}$$

$$= \frac{3}{10} - \frac{7u}{40}.$$

Now when $m=1$, u is OH, therefore $OH=-4$.

Therefore the focal length $F_1H = OH - OF_1 = -4 - 1\frac{5}{7} = -5\frac{5}{7}$ cms.

Hence $F_2H' = 5\frac{5}{7}$, but $O'F_2 = 4\frac{2}{7}$ cms.,

therefore $O'H' = 10$ cms.

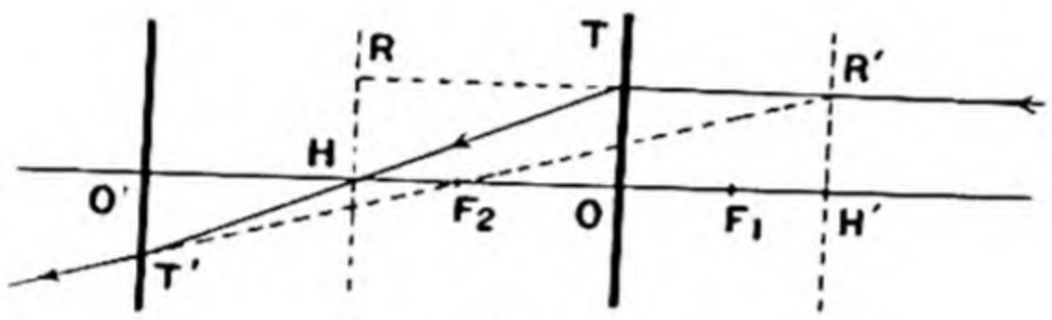

Fig. 62.

We are now able to make a diagram shewing the positions of the lenses O, O' and the principal foci and unit points.

If a ray strikes the first lens at right angles at T and the first unit plane at R, it must emerge in the direction $R'F_2$, where R' is on the second unit plane such that $H'R' = HR$. If $R'F_2$ cuts the second lens in T', the path of the ray between the lenses is TT' and the emergent ray is the production of $R'T'$.

As stated in Art. 85 F_1 is a real focus and F_2 is a virtual focus, and the combination is divergent for parallel rays falling first on the lens of focal length 4 cms., and convergent if the rays fall first on the lens of focal length 10 cms.

The focal length might of course have been obtained at once by the formula

$$\frac{1}{F} = \frac{1}{f} + \frac{1}{f'} + \frac{a}{ff'}.$$

92. Any number of thin lenses on the same axis.

Let $k_1, k_2 \ldots k_n$ denote the powers of the successive lenses, k_1 being the nearest to the incident light; and let $a_1, a_2 \ldots a_{n-1}$ denote the distances between them. Let u be the distance of a small object on the axis from the first lens O and v_1 the distance of its image from O; also let $v_2, v_3 \ldots v_{n-1}, v$ denote the distances of the

second, third, etc. images from the second, third, etc. lenses; v being the distance of the final image from the final lens, and suppose that all distances are positive when drawn to meet the incident light.

Then we have for the successive refractions

$$\frac{1}{v_1} - \frac{1}{u} = k_1$$

$$\frac{1}{v_2} - \frac{1}{v_1 + a_1} = k_2$$

$$\frac{1}{v_3} - \frac{1}{v_2 + a_2} = k_3$$

.....

$$\frac{1}{v} - \frac{1}{v_{n-1} + a_{n-1}} = k_n.$$

Whence $\frac{1}{v} = k_n + \frac{1}{a_{n-1} + v_{n-1}}$, or $v = \frac{1}{k_n + \frac{1}{a_{n-1} + v_{n-1}}}$,

and $\frac{1}{v_{n-1}} = k_{n-1} + \frac{1}{a_{n-2} + v_{n-2}}$, or $v_{n-1} = \frac{1}{k_{n-1} + \frac{1}{a_{n-2} + v_{n-2}}}$;

therefore $v = \frac{1}{k_n + \frac{1}{a_{n-1} + \frac{1}{k_{n-1} + \frac{1}{a_{n-2} + v_{n-2}}}}}$.

And continuing this process we get v given in terms of u by a continued fraction, viz.

$$v = \frac{1}{k_n + \frac{1}{a_{n-1} + \frac{1}{k_{n-1} + \frac{1}{a_{n-2} + }}}} \cdots\cdots + \frac{1}{a_1 + \frac{1}{k_1 + \frac{1}{u}}}.$$

If C/A and D/B denote the last but one and last but two convergents of this continued fraction, we have

$$v = \frac{Cu + D}{Au + B} \quad \text{.....(1).}$$

Also since C/A is an odd convergent, there being $2n$ in all,

therefore $$BC - AD = 1 \quad \text{.....(2).}$$

93. Linear magnification.

We have seen in Art. 78 that for a single thin lens the longitudinal magnification m' is the square of the transverse linear magnification m.

Now for refraction through n lenses

$$m' = m_1' m_2' m_3' \ldots m_n'$$

$$= m_1^2 m_2^2 m_3^2 \ldots m_n^2 = m^2,$$

where m, m' are the transverse and longitudinal magnifications due to the whole system and

$$m_1, m_2, \ldots m_n, \text{ and } m_1', m_2', \ldots m_n'$$

are the transverse and longitudinal magnifications due to the separate lenses.

Now $$m' = \frac{dv}{du} = \frac{C(Au+B) - A(Cu+D)}{(Au+B)^2} = \frac{BC - AD}{(Au+B)^2},$$

but $$BC - AD = 1;$$

therefore $$m = \pm \frac{1}{Au+B},$$

and from the special case of two lenses we know that the positive sign is correct, hence

$$m = \frac{1}{Au+B} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

It is not necessary to pursue the investigation of this case further, because formulae (1), (2), (3) are sufficient to determine the principal foci and unit points and they are the same as the corresponding formulae for two thin lenses in Arts. 79 and 81, hence the whole of the investigations of Arts. 79—89 apply to the case of n thin lenses.

EXAMPLES.

1. Two thin lenses both of focal length 15 cms. are on the same axis at a distance 10 cms. apart. Find the position and magnitude of the image of a small object 3 cms. in height placed at right angles to the axis 50 cms. from the nearer lens, (1) when both lenses are convergent, (2) when the nearer lens is divergent and the further convergent.

2. In Ramsden's eye-piece two convergent lenses of focal length f are at a distance $\frac{2}{3}f$ apart. Find the principal foci and unit points.

3. In Huyghens' eye-piece two convergent lenses are of focal lengths $3f$ and f at a distance $2f$ apart. Shew that the combination is convergent or divergent according as the light first falls on the first or second lens.

4. If a pair of thin lenses are convex and concave, each of numerical focal length f and at a distance $\frac{1}{2}f$ apart, give a diagram from which the positions of conjugate foci and the magnification can be found. (M. T. i. 1911.)

5. Find the planes of unit magnification for a system of two thin convex lenses, of focal lengths 2 inches and 6 inches respectively, at a distance of 4 inches from one another. (St John's Coll. 1906.)

6. If the focal lengths of two thin lenses on the same axis are 6 inches and 3 inches and the lenses convex and concave respectively, find when the system is convergent and when divergent. (Coll. Exam. 1908.)

7. A scale is placed at the first focal point of a lens A, and is viewed through A and a second lens B; prove that the image is formed at the second focal point of B, and that the magnification of the scale is equal to the ratio of the focal lengths of the lenses. Is the image erect or inverted?

(M. T. i. 1913.)

8. An object is viewed through two thin convex lenses of focal lengths f and f' placed at a distance d apart, the eye being close up to the latter. Prove that, if the distance of the image from the eye is Δ, the linear magnification is

$$1+\Delta\left(\frac{1}{f}+\frac{1}{f'}\right)-\frac{d}{f}\left(1+\frac{\Delta}{f'}\right).$$ (Coll. Exam. 1893.)

9. A convex lens of focal length f is placed in front of a luminous point P so that the image is formed at the shortest possible distance from P. If now a concave lens of very long focal length F be placed in contact with the first, shew that the image will be shifted a distance $\frac{4f^2}{F}$. (St John's Coll. 1884.)

10. The images of two small objects P, Q placed on the common axis of a system of lenses are p, q. Prove that the ratio of the distances pq, PQ is equal to the product of the magnifications of P, Q. (Coll. Exam. 1901.)

11. Shew that the size of an image, produced by an optical system whose foci and unit points are known, is increased by placing a plate of glass between the system and the object if the object is beyond the first focus of the system. (M. T. ii. 1911.)

12. An eye-piece is to be formed of two thin convex lenses, at distance a apart, and with its principal (or unit) points at distance c apart (reckoned numerically). Prove that, for different lenses fulfilling these conditions and such that the eye-piece is equivalent to a convex lens, the greatest focal length possible for the eye-piece is $(a+c)^2/4c$. (M. T. 1901.)

13. The erect image of a small object is thrown directly on a screen at distance c by two convergent lenses of numerical focal lengths f and f'. Prove that the extreme values of the magnification m are given by the quadratic equation

$$m+\frac{1}{m}-2=\frac{c-4f-4f'}{4ff'}c.$$ (M. T. 1909.)

14. Two thin lenses of focal lengths f and f' are placed on the same axis at a distance a apart. If a ray from a point on the axis in front of the first lens at a distance u from it be incident on this lens at a small distance y from the axis, shew that the deviation of the ray is

$$y\left(\frac{1}{f}+\frac{1}{f'}-\frac{a}{ff'}+\frac{a}{uf'}\right),$$

the focal length being taken positive for a convex lens. (Coll. Exam. 1902.)

15. A ray parallel to the axis is refracted through a thin convex lens, reflected at a plane mirror, and passes through the lens again. Find the equivalent lens of the system. (St John's Coll. 1897.)

16. If a system of two thin lenses is such that a single thin lens can be found which produces an image of any point on the axis in the same position as the system, shew that the distance between them must be twice the geometric mean of their focal lengths. (Coll. Exam. 1905.)

17. Two convex lenses of focal lengths $-a$, $-b$ are on the same axis at a distance $a+b$ apart. Shew that $u/a^2+v/b^2=1/a+1/b$ connects conjugate foci, where u, v are measured from the first and second lenses respectively, and estimated positively away from the system.

What change arises in case a third convex lens $(-c)$ be placed at a distance a from the first lens and b from the second?

(St John's Coll. 1906.)

18. Two thin coaxial lenses, focal lengths f_1, f_2, are at a distance apart equal to a. Prove (1) that if $a^2>4f_1f_2$ there are two positions of a point on the axis such that the point coincides with its conjugate; (2) that these positions are equidistant from the principal foci, and (3) that the magnifications in these two cases are the roots of the quadratic

$$m^2+m\left(\frac{a^2}{f_1f_2}-2\right)+1=0.$$ (Coll. Exam. 1901.)

19. Given in an optical system three fixed pairs of conjugate foci, give a geometrical construction to find the conjugate of any variable point.

(St John's Coll. 1897.)

20. Shew that there are two points on the axis of a symmetrical optical instrument such that each coincides with its own image and that their distance apart is $\sqrt{d^2+4dF}$, where F is the focal length of the instrument and d the distance apart of the principal points. (Coll. Exam. 1895.)

21. An optical system, consisting of any number of coaxial lenses, is so constituted that each of two points A, B is its own image. Prove that, if Q is any point and Q' its image, the magnification is

$$\sqrt{\frac{\overline{Q'A}\,.\,\overline{Q'B}}{\overline{QA}\,.\,\overline{QB}}}.$$ (Coll. Exam. 1905.)

22. Two thin lenses of focal lengths f_1 and f_2 are placed on the same axis at a distance a apart. Shew that, if a thin lens of focal length

$$-f_1f_2\,(f_1+f_2+a)/(f_1+f_2)^2$$

be placed between them so as to divide the distance in the ratio of the focal lengths f_1 and f_2, then the image of any object will be of the same size as the object and at a constant distance $a^2/(f_1+f_2+a)$ from it.

(Trinity Coll. 1891.)

23. A, B, C, D are four points taken in order on an axis such that the lengths of AB, BC, CD are 6, 6 and 8 inches respectively. At A, B, C are placed lenses of focal lengths 4, 9 and 6 inches, the middle lens being concave and the others convex: D is a luminous point. In what positions can a screen be placed perpendicular to the axis so that there may be a bright spot on it?

(M. T. 1902.)

24. Find the positions of the cardinal points for a system of three coaxial lenses, the intermediate lens being convex, and the outer lenses concave, with their foci coinciding at the centre of the intermediate lens. Notice particularly the case in which the numerical values of the three focal lengths taken in order are in harmonic progression, shewing that the effect is to produce an image of the same magnitude as the object, and at a constant distance from it. (Coll. Exam. 1894.)

25. Three convergent thin lenses, of focal lengths $4a$, a, and $4a$ respectively, are placed in order along an axis so that the distance between consecutive lenses is $4a$. Prove that this combination simply inverts every small object on the axis without change of magnitude or position.
(M. T. 1905.)

26. Two thin convergent lenses A, C of focal lengths F, G are placed at a distance $F+G$ apart. A third thin convergent lens B of focal length f is placed between them so that $AB=F$, $BC=G$, and the three lenses are coaxial. Shew that the magnification of any small object is independent of the position of the object and of the value of f and that the image is inverted.

Shew also that the image of any object point on the axis will be at a constant distance from the object if $F=G$ and that the distance will vanish if $F=G=4f$. (M. T. i. 1914.)

27. Find the power of the lens equivalent to four lenses of equal power, arranged at equal intervals along a common axis. What is the distance between four such lenses of given powers, when the power of the equivalent lens is a maximum or a minimum? (Coll. Exam. 1906.)

28. Four convex lenses, focal lengths a, b, b, a respectively, are placed on the same axis. Prove, by geometrical construction or otherwise, that if the distances between the first and second, and third and fourth lenses be each $a+b$, the linear magnitude of any object near the axis is equal to that of its image. Prove also that if the distance between the second and third lenses be $2b(a+b)/(a-b)$, a being $>b$, the positions of the object and image will coincide. (Coll. Exam. 1894.)

29. If β', β'', β''' are the magnitudes of the images after refraction through a system of lenses on the same axis for an object at distances ρ', ρ'', ρ''' from the first lens, shew that

$$\beta''\beta'''(\rho''-\rho''')+\beta'''\beta'(\rho'''-\rho')+\beta'\beta''(\rho'-\rho'')=0.$$

(Coll. Exam. 1895.)

30. $2n$ thin convex lenses, whose focal lengths are f_1, f_2 ..... f_{2n}, are placed on an axis so that the distance between any adjacent pair is the numerical sum of their focal lengths. Shew that the linear magnification of any small object placed anywhere on the axis and perpendicularly to it is

$$(-1)^n \cdot \frac{f_2 f_4 \ldots\ldots f_{2n}}{f_1 f_3 \ldots\ldots f_{2n-1}}.$$

(Coll. Exam. 1895.)

CHAPTER VII

THICK LENSES

REFRACTION THROUGH A NUMBER OF MEDIA SEPARATED BY COAXIAL SPHERICAL SURFACES

94. Direct incidence on a solid sphere of glass.

Before discussing the general thick lens, we shall consider the case in which the two refracting surfaces are portions of the same sphere. In this case and *in every case in which the spherical surfaces are concentric* it is convenient to use formulae involving distances from the centre of the sphere.

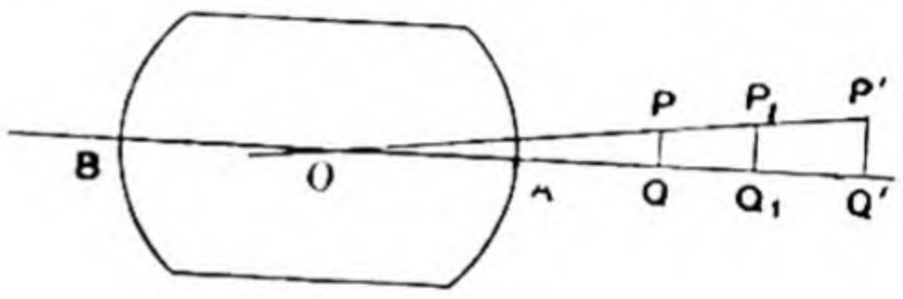

Fig. 63.

Let O be the centre of the sphere and μ the refractive index. If PQ is a small object represented by a straight line at right angles to the diameter OQ, the successive images P_1Q_1 formed by refraction at the surface A and $P'Q'$ formed by refraction at the surface B must be such that OQQ_1Q' and OPP_1P' are straight lines. In the figure the images are all placed on the right of O irrespective of their actual positions, in order that it may not be necessary to consider the question of their signs in writing down formulae. Thus, if r be the radius, by Art. 60(6) the distances are connected by

$$\frac{\mu}{OQ} - \frac{1}{OQ_1} = \frac{\mu - 1}{OA} = \frac{\mu - 1}{r},$$

and

$$\frac{1}{OQ_1} - \frac{\mu}{OQ'} = \frac{1 - \mu}{OB} = \frac{\mu - 1}{r}.$$

Therefore

$$\frac{1}{OQ} - \frac{1}{OQ'} = \frac{2(\mu - 1)}{\mu r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

We get the first principal focus F_1 as the limiting position of Q when Q' is at infinity; so that

$$OF_1 = \frac{\mu r}{2(\mu - 1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

In like manner the second principal focus F_2 is the limiting position of Q' when Q is at infinity; so that

$$OF_2 = \frac{-\mu r}{2(\mu - 1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Again the magnification $m = \dfrac{P'Q'}{PQ} = \dfrac{OQ'}{OQ}$.

For the unit points we require that m shall be unity or $OQ' = OQ$, and from (1) this is only possible when Q and Q' coincide at O. Therefore the unit points coincide at O.

If we give μ its approximate value 1·5 for glass, we find $OF_1 = 3r/2$ and $OF_2 = -3r/2$. The foci are therefore real foci.

Since the unit points H, H' coincide with O, the focal length F_1H is negative, being $-\mu r/2(\mu - 1)$.

The sphere acts as a convergent refracting system, and the equivalent thin lens would be a convergent lens of numerical focal length $\mu r/2(\mu - 1)$ placed in the position of the centre of the sphere.

95. Example. *A concave mirror is made of glass $\frac{1}{4}$ inch thick ($\mu = 1{\cdot}5$), silvered at the back, and the radius of curvature of the silvered surface is 2 feet. Find the position of the image of an object 6 ins. from the silvered surface and the magnification produced. Shew that the image is $\frac{3}{8}$ inch nearer the object than it would be if the glass were indefinitely thin and the silvered surface in the same position as before.* (M. T. 1913.)

If O is the centre of the surfaces and Q is the object, and OQ meets the surfaces in B and A, then with our usual convention about signs, $OA = -24$ ins. and $OQ = -18$ ins. and if we denote the thickness by t, $OB = -(24 - t)$ ins.

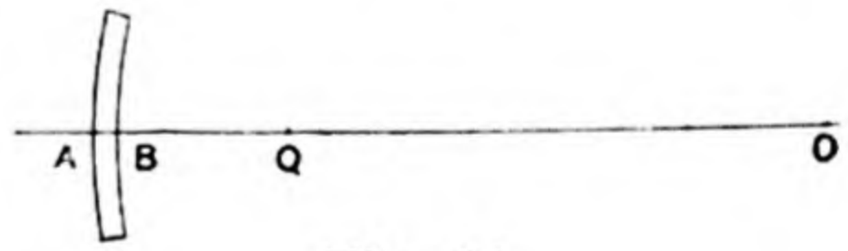

Fig. 64.

If Q_1, Q_2, Q' denote the images formed by refraction into the glass, reflection at the surface A and refraction out again at the surface B, we have

$$\frac{\mu}{OQ} - \frac{1}{OQ_1} = \frac{\mu - 1}{OB}, \quad \text{(Art. 60 (6))},$$

$$\frac{1}{OQ_1} + \frac{1}{OQ_2} = \frac{2}{OA}, \quad \text{(Art. 51 (4))}.$$

For the refraction out from the glass into air with the light now proceeding from left to right, if we regard the path as reversed in direction, Q' and Q_2 will still be conjugate points and we may take it that light travelling from right to left, proceeding from Q', after refraction into glass, has a geometrical focus at Q_2, hence

$$\frac{\mu}{OQ'} - \frac{1}{OQ_2} = \frac{\mu - 1}{OB}, \quad \text{(Art. 60 (6))}.$$

By adding these equations, we get

$$\frac{\mu}{OQ'} + \frac{\mu}{OQ} = \frac{2(\mu-1)}{OB} + \frac{2}{OA};$$

or, putting in the values,

$$\frac{1{\cdot}5}{OQ'} - \frac{1}{12} = \frac{1}{-24+t} - \frac{1}{12}.$$

Therefore $$OQ' = -1{\cdot}5(24-t).$$

Hence if $t=\frac{1}{4}$, $OQ' = -35\frac{5}{8}$ ins.

and if $t=0$, $OQ' = -36$ ins.;

shewing that Q' is nearer to Q by $\frac{3}{8}$ in. in the former case than in the latter.

For the magnification we have, since the successive images of the same point must lie on a straight line through O,

$$m = \frac{OQ'}{OQ} = \frac{35\frac{5}{8}}{18} = \frac{95}{48}.$$

96. Examples. 1. A magnifier is formed of a glass sphere 2·5 cm. in diameter, for which $\mu=1{\cdot}5$.

Find the positions of the image of an object whose distance from the surface of the sphere is (1) ·5 cm., (2) ·625 cm., (3) 1 cm. Also find the linear magnification in each case.

(Coll. Exam. 1911.)

2. Prove that a sphere of refractive index $\frac{4}{3}$ produces virtual images of small objects whose distances from its surface are less than its radius.

(Coll. Exam. 1904.)

3. Prove that if a pencil proceeding from a point passes directly through a sphere, the incident and emergent parts of each ray near the axis intersect on the same plane through the centre of the sphere at right angles to the axis.

(Coll. Exam. 1903.)

4. An object is seen by direct vision through a sphere of radius $\frac{1}{2}$ inch and refractive index 1·5. If the eye be placed at a distance of $\frac{3}{4}$ inch from the centre of the sphere, find where the object must be placed in order that the image may be 12 inches from the eye.

(M. T. i. 1910.)

5. Prove that the focal length of a spherical lens is to that of a hemispherical lens of the same radius as half the index of refraction is to unity.

(St John's Coll. 1896.)

6. A thick glass globe, of inner and outer radii 9 and 12 inches respectively, is filled with water. Find the position and magnitude of the images of specks on the glass, as seen from the opposite side. (For glass $\mu=3/2$, for water $\mu=4/3$.)

(Coll. Exam. 1910.)

7. A magnifier is formed of a glass sphere 2 cm. in diameter, for which $\mu=1{\cdot}5$. How far from the surface should an object be placed so as to obtain an image 25 cm. from the eye, which is placed close up to the sphere, on the opposite side from the object? Find also the linear magnification.

(St John's Coll. 1909.)

8. A small bright object is in front of a sphere of glass of radius a and refractive index μ behind which is a concentric spherical mirror of radius b. Rays from the object are refracted through the sphere, reflected by the mirror and refracted through the sphere again. Shew that an image of the same size and in the same position would be formed by a mirror of radius R where $\frac{1}{R} = \frac{2}{a}\left(1 - \frac{1}{\mu}\right) - \frac{1}{b}$; and shew how the mirror must be placed, and which side must be silvered, according as the distance of the object from the centre of the sphere is greater or less than R, and according as R is positive or negative. (M. T. 1903.)

9. A sphere of radius r is silvered on the back surface. A pencil issuing from an origin at a distance p from its centre is incident directly upon it, and after one refraction and $(2n-1)$ internal reflections is refracted out at the face by which it entered. If q_{2n-1} be the distance of the focus of the emergent pencil, prove that

$$\frac{1}{q_{2n-1}} - \frac{1}{q_{2n+1}} = \frac{4}{\mu r}.$$ (St John's Coll. 1884.)

10. A spherical globe of glass is filled with water. A pencil of parallel rays is incident upon it. Find the geometrical focus after passing through the glass. If the external radius is R, and the absolute refractive indices of the glass and water are μ and μ', and if the thickness t of the glass is small, shew that the distance of the geometrical focus from the centre of the globe is approximately

$$\frac{\mu' R}{2(\mu' - 1)} + \frac{\mu'(\mu - \mu')\, t}{2\mu(\mu' - 1)^2}.$$ (Coll. Exam. 1902.)

11. A hollow sphere of glass (index $\frac{3}{2}$) has its internal radius 2 inches and its external radius 3 inches, and is filled with water (index $\frac{4}{3}$). A small piece of metal rests at the lowest point of the water, and is viewed by an eye situated vertically above the centre of the sphere. Shew that the image seen is greater than the object in the ratio 18 : 11, and determine its position. (St John's Coll. 1906.)

97. A thick lens.

The most general case of a lens bounded by spherical surfaces that we can consider is that in which there are three different media separated by two spherical surfaces, corresponding to a lens with different media on opposite sides of it.

Let μ_0, μ_1, μ_2 be the refractive indices, r, s the radii of the spheres, and T the distance between the surfaces measured along the axis or line of centres, *i.e.* T is the thickness of the lens.

Let the surfaces cut the axis in O, O' and let Q, Q_1, Q' be a point on the axis and its successive images.

Let $OQ = U$, $OQ_1 = V_1$ and $O'Q' = V$, all counted as positive when drawn to meet the incident light which is assumed to come from right to left in accordance with our usual convention. The radii r and s measured from O and O' are positive when the surfaces are concave to the incident light.

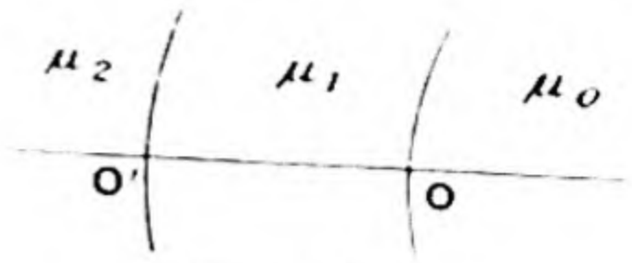

Fig. 65.

The formulae for the successive refractions are

$$\frac{\mu_1}{V_1} - \frac{\mu_0}{U} = \frac{\mu_1 - \mu_0}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and

$$\frac{\mu_2}{V} - \frac{\mu_1}{V_1 + T} = \frac{\mu_2 - \mu_1}{s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If we put $U = \mu_0 u$, $V_1 = \mu_1 v_1$, $V = \mu_2 v$ and $T = \mu_1 t$, then u, v_1, v are called "reduced distances" and t is the reduced thickness; and if we denote the powers of the refracting surfaces by k and k', we may write the last equations

$$\frac{1}{v_1} - \frac{1}{u} = k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and

$$\frac{1}{v} - \frac{1}{v_1 + t} = k' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

As in Art. 79, these lead to

$$v = \frac{u(tk+1) + t}{u(tkk' + k + k') + tk' + 1},$$

or

$$v = \frac{Cu + D}{Au + B} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

where

$$\left.\begin{aligned} A &= tkk' + k + k', \quad B = tk' + 1 \\ C &= tk + 1, \qquad\quad D = t \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(6),$$

and

$$BC - AD = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

98. Principal Foci.

The first principal focus F_1 is the limiting position of Q when Q' is at infinity. Putting $v = \infty$, we get $u = -B/A$, and therefore

$$OF_1 = -\mu_0 \frac{B}{A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The second principal focus F_2 is the limiting position of Q' when Q is at infinity. Putting $u = \infty$, we get $v = C/A$, and therefore

$$O'F_2 = \mu_2 \frac{C}{A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If Q, Q' are any pair of conjugate foci,

$$F_1Q \,.\, F_2Q' = (OQ - OF_1)\,(O'Q' - O'F_2)$$
$$= \mu_0\mu_2\left(u + \frac{B}{A}\right)\left(v - \frac{C}{A}\right),$$

or, as in Art. 80,

$$F_1Q \,.\, F_2Q' = -\frac{\mu_0\mu_2}{A^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Hence conjugate points lie on opposite sides of corresponding principal foci, and if one is moved the other moves in the same direction.

99. Linear magnification produced by a thick lens.

Since, by Art. 62, the linear magnification produced by a single refraction at a spherical surface is the ratio of the reduced distances of the image and object from the surface, therefore

$$\frac{1}{m} = \frac{u}{v_1} \cdot \frac{v_1 + t}{v},$$

and, as in Art. 81, this may be reduced to

$$\frac{1}{m} = Au + B \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

or

$$m = C - Av \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The **unit points** H, H' of the lens are obtained by putting $m = 1$ in these last results. Hence we have

$$OH = \mu_0 u = \mu_0 \frac{1 - B}{A} \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and

$$O'H' = \mu_2 v = \mu_2 \frac{C - 1}{A} \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

From (3) and (4) combined with (1) and (2) of the last article we get

$$F_1H = OH - OF_1 = \frac{\mu_0}{A},$$

and

$$F_2H' = O'H' - O'F_2 = -\frac{\mu_2}{A}.$$

The lengths F_1H and $H'F_2$ are therefore in the ratio $\mu_0 : \mu_2$, and are equal only when the medium on both sides of the lens is the same. They are called the **focal lengths** of the lens for the given media and may be denoted by f_1, f_2. Equation (3) of the last article may now be written

$$F_1Q \,.\, F_2Q' = F_1H \,.\, F_2H' = -f_1f_2.$$

100. The construction for the image of a small object on the axis, given in Art. 86, will clearly hold in the case of a thick lens, with the reservation that F_1H, $H'F_2$ are not equal unless the media on opposite sides of the lens are the same.

The argument of Art. 87, that incident rays diverging from or converging to a point on the first focal plane emerge as a parallel pencil also clearly holds good.

101. The distance between the unit points.

We have
$$H'H = T + OH - O'H'$$
$$= T + (2 - B - C)/A.$$

For a lens in air this becomes
$$H'H = T - \frac{t(k+k')}{k+k'+tkk'}$$
$$= T - \frac{t(s-r)}{s-r-t(\mu-1)};$$

and if the thickness T or μt be small compared to $s-r$, we have approximately
$$H'H = T - t$$
$$= T\left(1 - \frac{1}{\mu}\right).$$

Thus for a thin lens of refractive index 1·5,
$$H'H = \tfrac{1}{3}T \text{ approximately.}$$

102. Different forms of lenses.

With the foregoing notation, we have for the focal length f of a lens of refractive index μ in air
$$\frac{1}{f} = (\mu-1)\left(\frac{1}{r} - \frac{1}{s}\right) - \frac{T}{\mu}\frac{(\mu-1)^2}{rs};$$
and the positions of the cardinal points are given by
$$OH = \frac{rT}{\mu(s-r) - (\mu-1)T}, \quad OF_1 = \frac{-r\{\mu s - T(\mu-1)\}}{(\mu-1)\{\mu(s-r) - (\mu-1)T\}}$$
and
$$O'H' = \frac{sT}{\mu(s-r) - (\mu-1)T}, \quad O'F_2 = \frac{s\{\mu r + T(\mu-1)\}}{(\mu-1)\{\mu(s-r) - (\mu-1)T\}}.$$

Also
$$H'H = \frac{(\mu-1)(s-r-T)T}{\mu(s-r) - (\mu-1)T}.$$

I. *Double-convex lens.*

In this case r is negative, and the focal length is positive or negative according as

$$T > \text{or} < \frac{\mu}{\mu - 1}(s - r).$$

In the former case OH and OF_1 are positive and $O'H'$ and $O'F_2$ are negative, *i.e.* the cardinal points are all outside the lens and the foci lie between the unit points. This is however a very exceptional case, for the condition implies that the thickness is about or more than three times the sum of the radii.

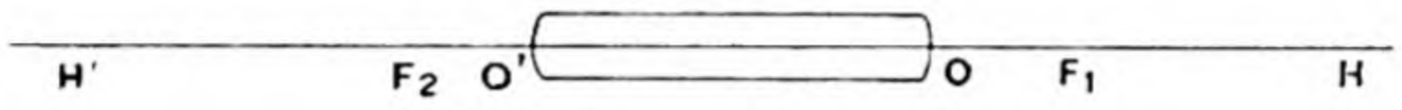

Fig. 66.

In the ordinary case, when $T < \frac{\mu}{\mu - 1}(s - r)$, the unit points fall within the lens and the foci outside as in Fig. 67, though the relative positions of the points H, H' will be as in the figure or interchanged according as $s - r$ is $>$ or $< T$.

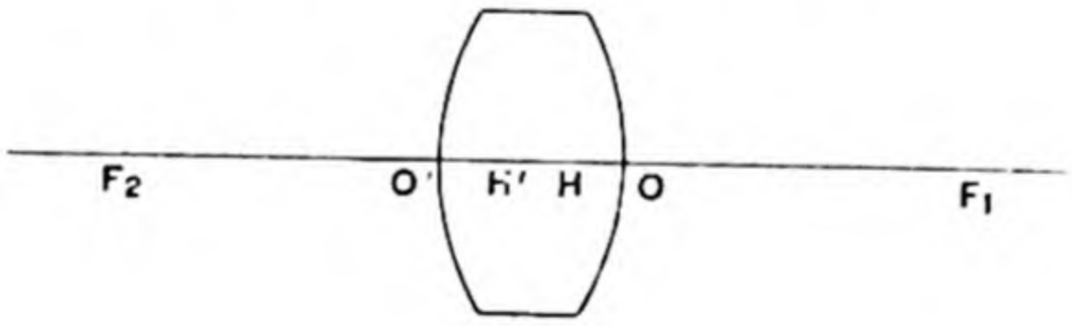

Fig. 67.

II. *Double-concave lens.*

In this case s is negative, and the focal length is positive; H, H' fall within the lens, $H'H$ is positive, and the foci are as in Fig. 68.

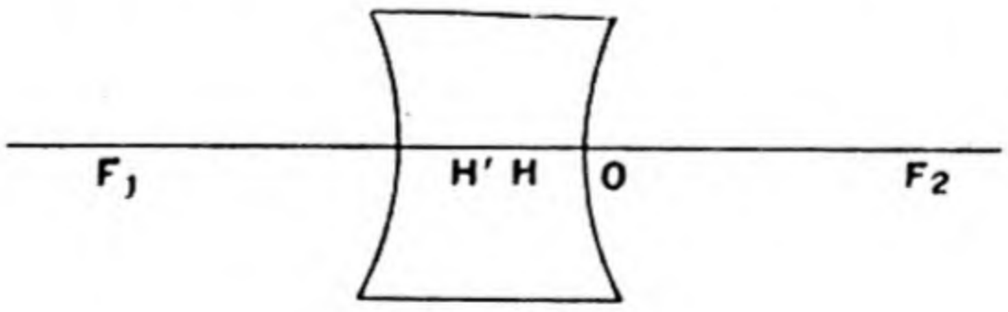

Fig. 68.

III. *Meniscus.*

A meniscus lens has one side concave and the other convex. Since the power of a lens is not altered by reversing it, it will be sufficient to consider the case of r and s both positive.

If $r > s$, the focal length is negative. The second surface is more curved than the first. OH, $O'H'$ and $O'F_2$ are negative and OF_1 is positive unless we have an exceptional case in which $T > \mu s/(\mu - 1)$. The general form is shewn in Fig. 69.

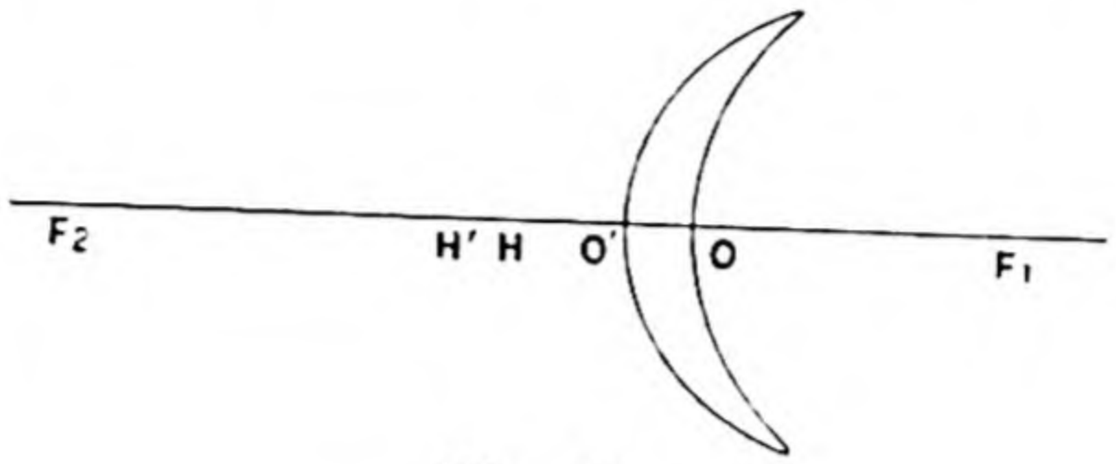

Fig. 69.

If $r < s$, the focal length is positive or negative according as $T <$ or $> \dfrac{\mu}{\mu - 1}(s - r)$.

In the former case OH, $O'H'$, and $O'F_2$ are positive and OF_1 is negative. The general form is shewn in Fig. 70.

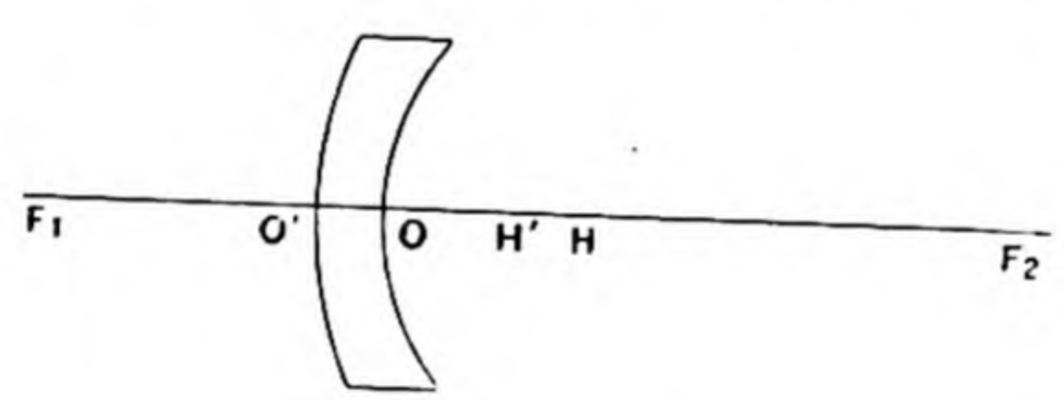

Fig. 70.

$H'H$ is positive or negative according as $s - r >$ or $< T$, and the foci F_1, F_2 are generally at a considerable distance.

In the latter case OH, $O'H'$ are negative, but this is not a case of any importance.

IV. *Lens with a plane face.*

When the first surface is plane $r = \infty$ and $O'H'$ is zero, so that a unit point is on the curved surface. Also $H'H = \dfrac{\mu - 1}{\mu} T$ so that H also falls within the lens.

103. Nodal points.

There is a fixed pair of conjugate points on the axis of a lens such that every ray incident through the first emerges through the second parallel to its original direction. These are called the **nodal points** of the lens.

The nodal points may be found by a geometrical construction as follows:

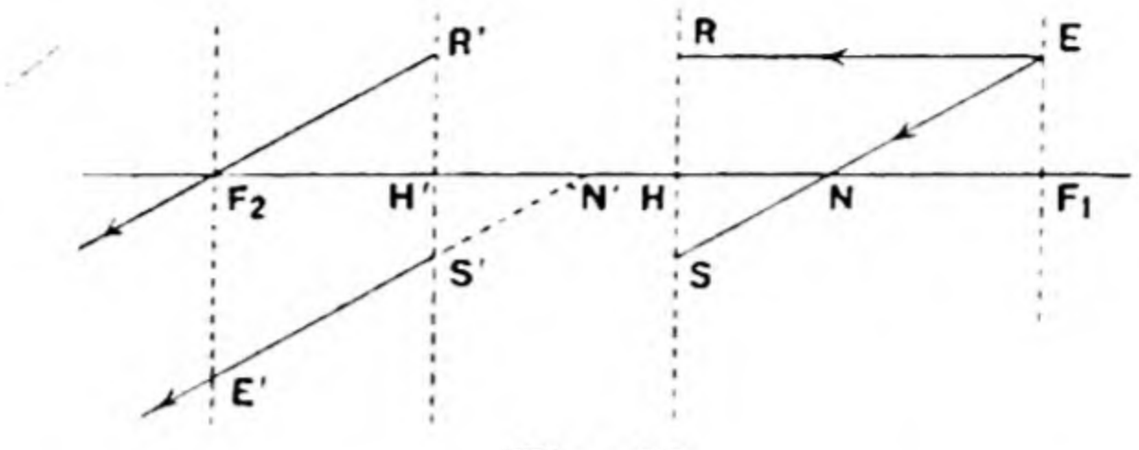

Fig. 71.

Let a ray parallel to the axis cut the first focal plane in E and the first unit plane in R. The emergent ray is then $R'F_2$, where $H'R' = HR$. Draw an incident ray ENS parallel to $R'F_2$ and cutting the axis in N and the first unit plane in S. The emergent ray cuts the second unit plane in S' where $H'S' = HS$ and is parallel to $R'F_2$, because the two incident rays emanate from a point E on the first focal plane. Therefore $S'E'$ is parallel to ENS and if it cuts the axis in N', N and N' are the nodal points. That they are fixed points is clear from the figure, for the triangles EF_1N and $R'H'F_2$ are identically equal so that $F_1N = H'F_2$; and if $N'S'E'$ cuts the second focal plane in E', $F_2E' = R'S' = RS$, so that the triangles $E'F_2N'$ and SRE are identically equal, whence we get $F_2N' = HF_1$.

If the media on both sides of the lens are the same, so that the focal lengths are equal, it is clear that the nodal points N, N' coincide with the unit points H, H'.

104. We may also obtain the nodal points by using Helmholtz's formula for linear magnification. Thus if l_0, l_1, l_2 denote an object and its successive images and α_0, α_1, α_2 the angles of divergence of a ray crossing the axis at the positions of the object and the successive images, we have

$$\mu_2 l_2 \alpha_2 = \mu_1 l_1 \alpha_1 = \mu_0 l_0 \alpha_0.$$

For nodal points $\alpha_2 = \alpha_0$,

therefore $$\mu_2 l_2 = \mu_0 l_0.$$

But $l_2/l_0 =$ linear magnification $= 1/(Au + B) = C - Av$;

therefore $\mu_2 = \mu_0 (Au + B)$, and $\mu_2 (C - Av) = \mu_0$.

Hence $$ON = \mu_0 u = \frac{\mu_2 - \mu_0 B}{A}, \text{ but } OF_1 = -\frac{\mu_0 B}{A},$$

therefore $$F_1 N = \frac{\mu_2}{A} = H'F_2.$$

Also $$O'N' = \mu_2 v = \frac{\mu_2 C - \mu_0}{A}, \text{ but } O'F_2 = \frac{\mu_2 C}{A},$$

therefore $$F_2 N' = -\frac{\mu_0}{A} = HF_1.$$

It follows that $NN' = HH'$; and for a thin lens the nodal and unit points coincide with the intersection of the lens with the axis.

105. Optical centre of a lens.

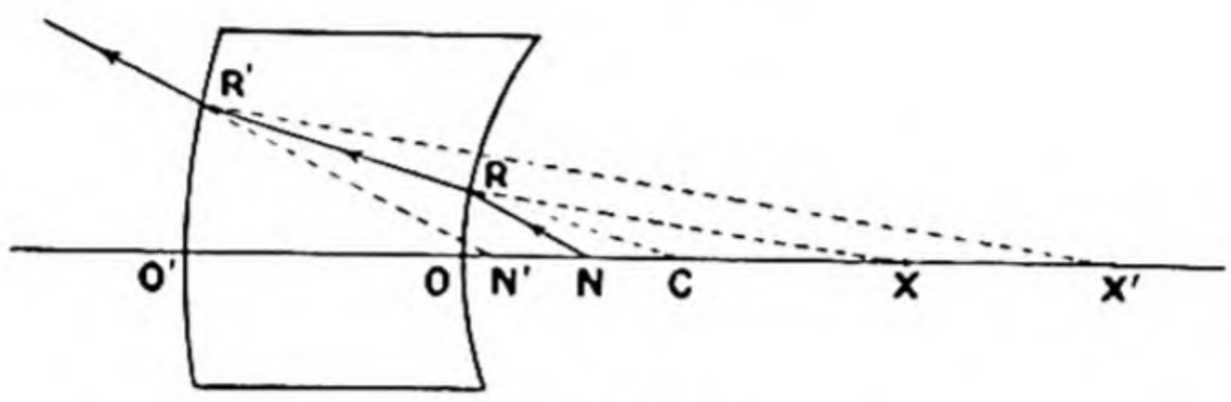

Fig. 72 (i).

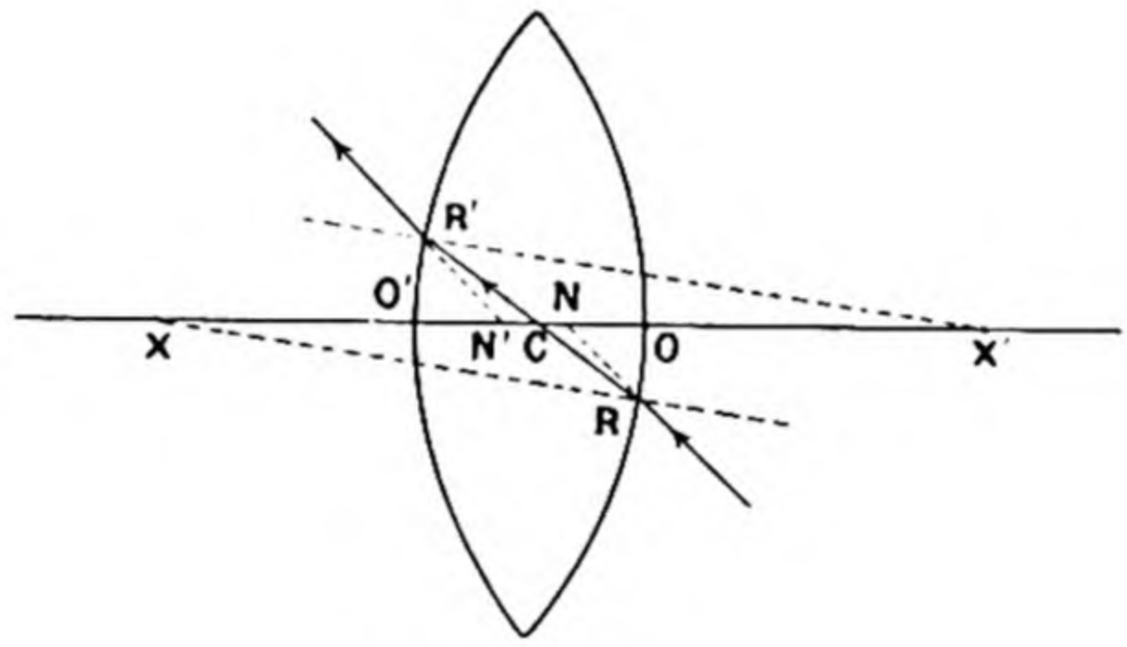

Fig. 72 (ii).

Let X, X' be the centres of the spherical surfaces of a lens with the same medium on both sides of it, and draw parallel radii XR, $X'R'$. Let RR' cut the axis in C. Since the tangent planes at R, R' are parallel, a ray whose path in the glass is RR' will undergo the same deflection at R and R', hence any ray whose path in the glass passes through the point C has its incident and emergent parts parallel. Again C is a fixed point, because by similar triangles

$$\frac{CX}{CX'} = \frac{CR}{CR'} = \frac{OC}{O'C}, \text{ therefore also } = \frac{OX}{O'X'},$$

or $$OC : O'C = r : s.$$

C is called the **optical centre** of the lens.

Also if N, N' are the points conjugate to C with regard to the first and second refracting surfaces, a ray incident through N passes

through C after the first refraction and on emergence passes through N' parallel to its original direction. Therefore N, N' are the nodal points.

Hence for a lens in air the unit points (which coincide in this case with the nodal points) are the points conjugate to the optical centre with regard to the two refracting surfaces.

106. Linear magnification due to a thick lens.

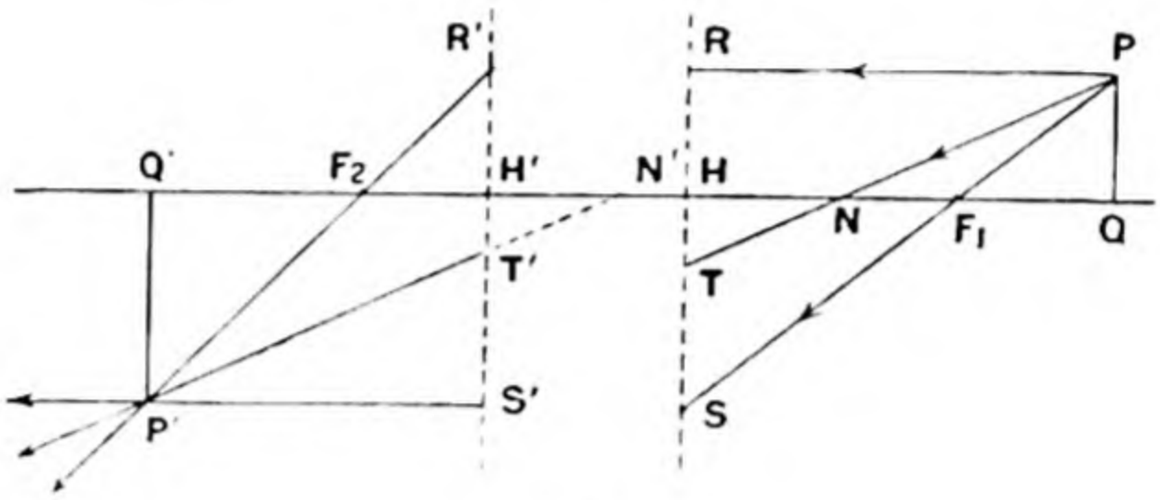

Fig. 73.

If we make the usual construction for the image of a small object PQ by means of the principal foci and unit planes; and also draw a ray PNT through the first nodal point N cutting the first unit plane in T, the emergent ray $N'T'P'$ passes through the second nodal point N' and cuts the second unit plane at T' where

$$H'T' = HT.$$

Then we have

$$m = \frac{P'Q'}{PQ} = \frac{SH}{PQ} = \frac{F_1H}{F_1Q};$$

also

$$= \frac{P'Q'}{R'H'} = \frac{F_2Q'}{F_2H'};$$

also

$$= \frac{N'Q'}{NQ} \quad \text{by similar triangles;}$$

also

$$= \frac{H'Q' \times P'H'Q'}{HQ \times PHQ},$$

but by Helmholtz's theorem since an object and image at H and H' are equal

$$\mu_0 PHQ = \mu_2 P'H'Q'.$$

Therefore

$$m = \frac{H'Q'/\mu_2}{HQ/\mu_0}.$$

107. Example. *Find the positions of the principal planes and principal foci of a hemispherical lens whose refractive index is* 3/2. (M. T. 1912.)

As explained in Art. 90 it is not necessary to remember all the formulae we have established for a thick lens in order to work out a numerical example of this kind.

If PQ is a small object in front of the plane face, an equal image P_1Q_1 is formed by refraction at the plane face, and $OQ_1 = \mu OQ$.

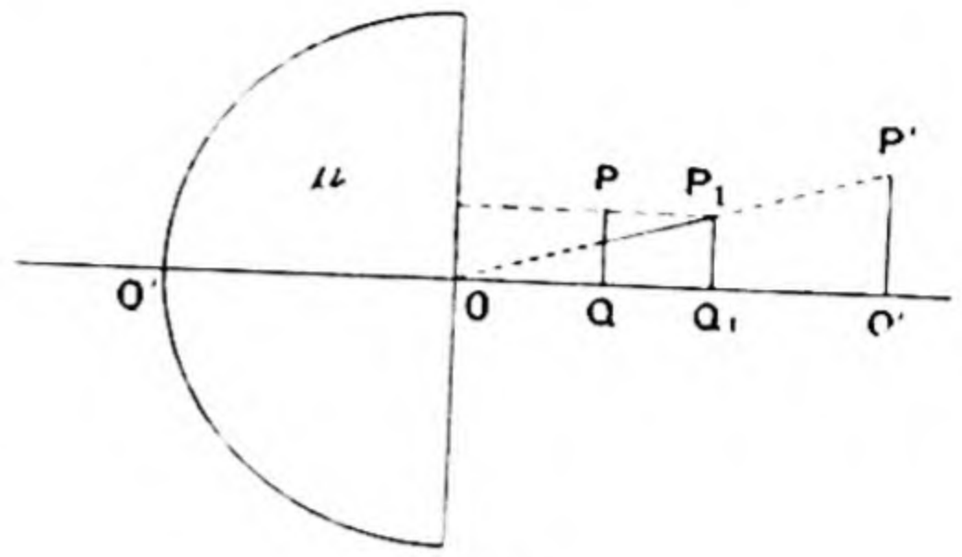

Fig. 74.

A refraction then takes place at the curved surface and if $P'Q'$ is the final image, using the formula (6) of Art. 60

$$\frac{1}{OQ_1} - \frac{\mu}{OQ'} = \frac{1-\mu}{OO'} = \frac{1-\mu}{-r},$$

if r is the radius of the hemisphere. Therefore

$$\frac{1}{\mu OQ} - \frac{\mu}{OQ'} = \frac{\mu - 1}{r},$$

or

$$\frac{2}{3OQ} - \frac{3}{2OQ'} = \frac{1}{2r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

When Q' is at infinity Q is at F_1, therefore $OF_1 = \frac{4}{3}r$; also when Q is at infinity Q' is at F_2, therefore $OF_2 = -3r$. Again the linear magnification m

$$= \frac{P'Q'}{PQ} = \frac{P'Q'}{P_1Q_1} = \frac{OQ'}{OQ_1} = \frac{OQ'}{\mu OQ}.$$

Hence for the unit planes, since $m = 1$, we must have

$$OH' = \mu OH \text{ or } 2OH' = 3OH,$$

but from (1)

$$\frac{2}{3OH} - \frac{3}{2OH'} = \frac{1}{2r}.$$

Therefore

$$OH = -\tfrac{2}{3}r \text{ and } OH' = -r.$$

108. Examples. 1. Shew that the focal length of a double convex lens (of radii r, s and thickness t) is equal to $rs/(\mu - 1)\,c$, and that the distance of the first unit point from the first face is $rt/\mu c$, where

$$c = r + s - (1 - 1/\mu)\,t.$$

Calculate the focal lengths and draw diagrams to indicate the relative positions of the unit and focal points in the following lenses:

(1) $r = 4$, $s = 6$, $t = 1$, $\mu = 1{\cdot}5$,

(2) $r = 4$, $s = -6$, $t = 1$, $\mu = 1{\cdot}5$.

(St John's Coll. 1910.)

2. Calculate the positions of the unit points and principal foci of a thick double-convex lens, the radii of whose surfaces are 4 cm. and 3 cm., the thickness of the lens being 2·5 cm., and the refractive index 1·5, and draw a figure shewing the path of a ray. (Coll. Exam. 1912.)

3. Obtain the necessary radii for the surfaces of a converging lens of thickness 1 in. and refractive index 1·5 in order that the focal length may be 6 ins., and that one principal focus may be at a distance of 5 ins. from the nearer surface of the lens. (St John's Coll. 1908.)

4. A pencil of radius y is incident parallel to the axis on a thick lens; prove that with the ordinary notation the angle of divergence of the emergent pencil is $(\mu-1)\left\{\frac{1}{r}-\frac{1}{s}-\frac{(\mu-1)\,t}{\mu rs}\right\}y$, and that the coefficient of y is the power of the lens. (Trinity Coll. 1890.)

5. A thick convex lens bounded by spherical surfaces, each of radius a, is such that the centres of the surfaces lie outside the lens. Prove that, if $\mu=d/(d-a)$, then the distance between the principal points of the lens is $d-4f$, where d is the distance apart of the centres and f is the numerical focal length of the lens. (Coll. Exam. 1903.)

6. If one face of a lens is plane, shew that one principal point is on the curved face, and find the other principal point.

Find the magnification of a small object close to the plane face in terms of the radius of the curved face and the thickness of the lens.

(Coll. Exam. 1913.)

7. The thickness t of a convex lens is small compared with its focal length, the numerical value of which is ϕ. Shew that the least distance between an object and its real image formed by the lens is approximately

$$4\phi+\frac{\mu-1}{\mu}t.$$ (Coll. Exam. 1902.)

8. Prove that when the distance between an object and its real image formed by a thick convex lens is a minimum, the image is inverted and of the same size as the object. (Coll. Exam. 1900.)

9. If a lens be double-concave of radius r, and the object at distance r from the first surface, prove that the error in m the magnifying power made by taking the lens as thin is $\frac{m^2t}{r}(\mu-1)$. (Coll. Exam. 1888.)

10. Prove that if κ_1 and κ_2 are the powers of the two surfaces of a lens, and t its thickness, the condition that the two foci coincide is

$$\kappa_1\kappa_2t^2+(\mu-1)(\kappa_1+\kappa_2)t-2\mu=0,$$

where μ is the index of refraction. (Trinity Coll. 1910.)

11. Two hemispherical lenses whose refractive index is $\frac{3}{2}$ are placed on the same axis with their curved surfaces in contact. Prove that the focal length of the combination is equal to the radius. (M. T. i. 1912.)

12. Three media are separated by two spherical interfaces on a common axis. Prove geometrically that the first principal focus for refraction at the first interface only and the second principal focus for refraction at the second interface only are conjugate points. (Trinity Coll. 1904.)

13. Find the positions of the principal planes, nodal points, and principal foci for a double-convex glass lens of thickness t, whose faces are portions of spheres of radius ρ, let into the side of a vessel full of water and used for viewing objects in the water. (St John's Coll. 1904.)

14. Shew that for a sphere of glass of radius a and index μ, with media of index μ_1 and μ_2 on the two sides of it, the values of the focal lengths are

$$a\mu_1/(\mu_1+\mu_2-2\mu_1\mu_2/\mu) \text{ and } a\mu_2/(\mu_1+\mu_2-2\mu_1\mu_2/\mu).$$

(St John's Coll. 1897.)

15. A point of light is distant 22 inches from a screen. A lens directly between the two throws an image on the screen when in either of two positions 6 inches apart. If the light is moved to 29 inches from the screen, the two positions of the lens are 15 inches apart. Calculate the focal length of the lens, and shew that its focal points are one foot apart. (Coll. Exam. 1906.)

109. Experimental determination of the focal length of a thin lens.

First *let the lens be convergent.* We know that if f be the numerical focal length of a thin lens, and an object be placed at a distance $2f$ from the lens, a real inverted image of the same magnitude is formed at a distance $2f$ from the lens on the opposite side.

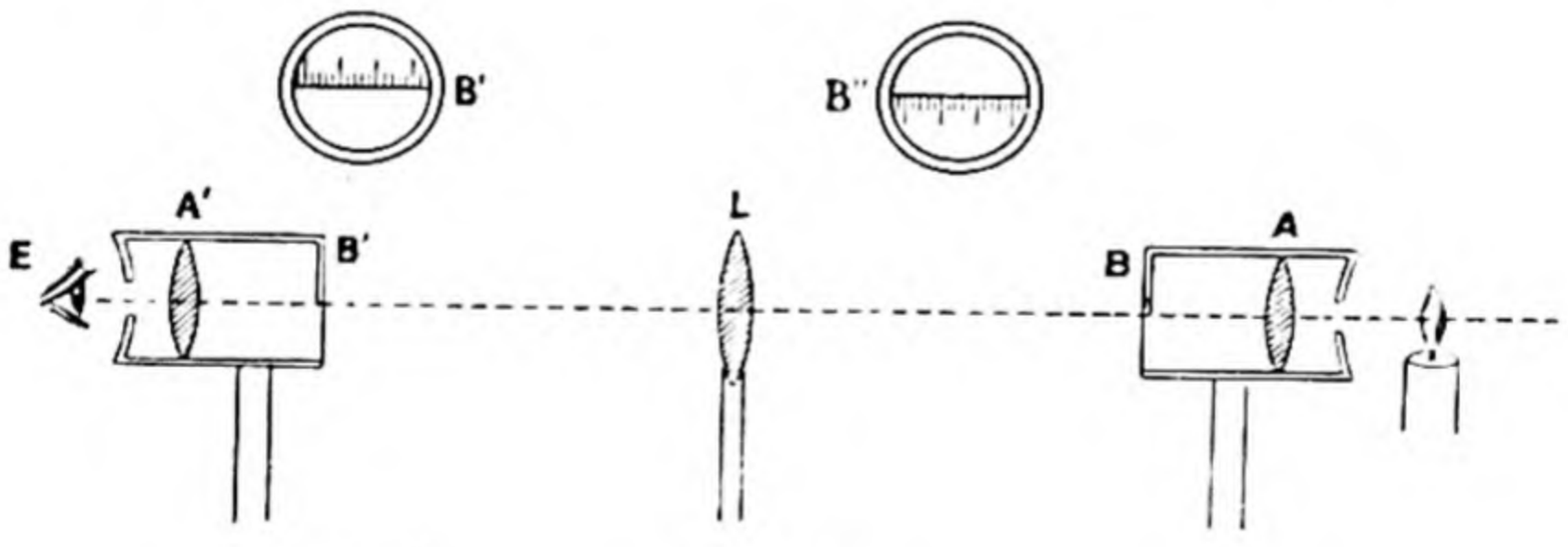

Fig. 75.

The lens L is mounted on a stand the foot of which can slide along a horizontal graduated ruler. B, B' are two similar semi-discs of ivory or ground glass with millimetre subdivisions marked upon them. Light is thrown upon B by a lens A and another lens A' is adjusted on the other side of B' so that the divisions on B' are clearly visible to an eye at E. The relative positions of the pieces are then adjusted until the image of B, which presents the appearance B'', is found filling up the lower half of the circle B', the division marks on B'' coinciding with the productions of the division marks on B'. The distance BB' is then $4f$, or to a closer approximation $4f+\frac{1}{3}t$, where t is the thickness of the lens (Art. 101).

110. A second and somewhat similar method is based on the fact that the minimum distance between an object and its real image formed by a convergent lens of numerical focal length f is $4f$.

To prove this, if x be the distance between object and image, in the formula

$$\frac{1}{v} - \frac{1}{u} = -\frac{1}{f}$$

we have $$v = -(x - u);$$

so that $$\frac{1}{x-u} + \frac{1}{u} = \frac{1}{f}$$

or $$u^2 - ux + fx = 0,$$

and for a real value of u we must have

$$x^2 \not< 4fx$$

or $$x \not< 4f.$$

111. Secondly, *when the lens is divergent*, it may be placed in contact with a thin convergent lens of such power that the combination is convergent. The power of the combination can be determined by the foregoing methods, and also that of the convergent lens and then that of the divergent lens follows from the fact that the power of the combination is the algebraical sum of the powers of the two lenses.

112. To determine experimentally the positions of the cardinal points of a system of lenses.

Assuming the system to be a convergent one, the distances from the lenses of the principal foci F_1, F_2 can be found by bringing parallel rays from a distant source of light to a focus on a screen and measuring the distance of the screen from the nearest lens.

To find the unit points H, H' we may observe that if the initial and final media are the same, they coincide with the nodal points N, N'. If a pencil

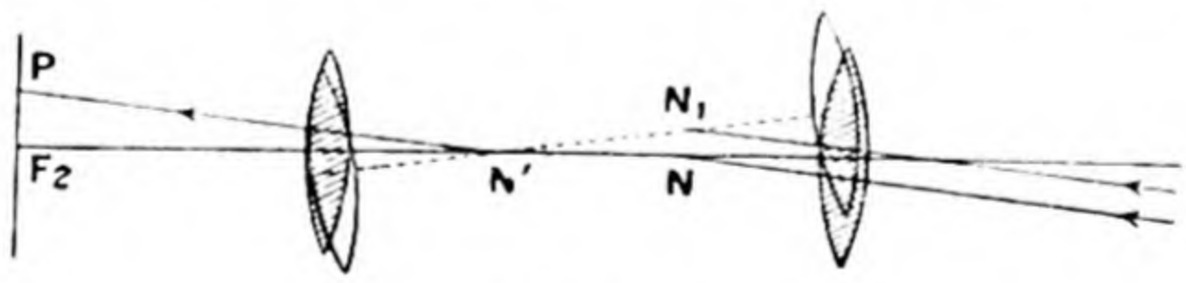

Fig. 76.

of parallel rays from a distant source of light be brought to a focus at a point P on a screen in the second focal plane, that one of the rays that is incident through the first nodal point N emerges parallel to its original direction in the line $N'P$. If the system of lenses undergoes a small rotation about the point N', one of the incident parallel rays will pass through N_1, the new position of N, and will emerge in the same line $N'P$ as the former ray. Hence the position of the point P on the screen will not be altered by small rotations of the system about the point N'. This gives a practical method of finding the position of N'. The combination is mounted on a "nodal slide," that is a carriage sliding on rails, which rails can turn about a vertical axis. By adjusting the carriage the nodal point of the system can be brought into the vertical axis of rotation. Whenever it is found that a small rotation does not alter the position on the screen of the image of a distant object, then N' is on the axis of rotation.

113. Another method*.

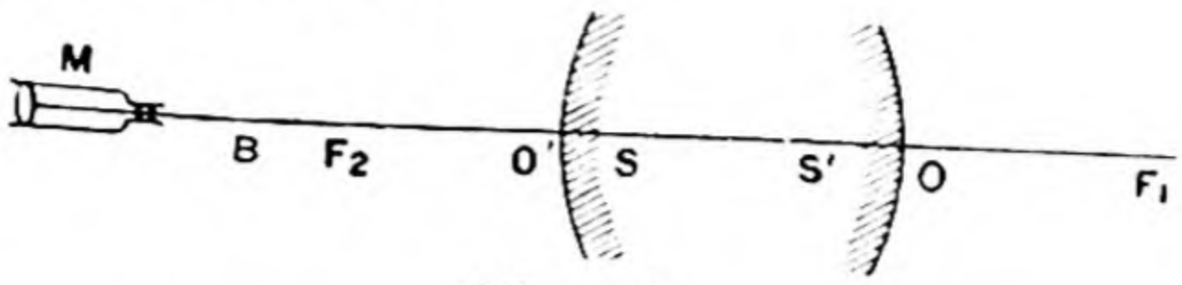

Fig. 77.

This method determines the focal length from the formula for the distances of conjugate points from the principal foci, viz. $F_1Q \,.\, F_2Q' = -f^2$.

A microscope M of long focal length is used for observing the positions of images. It must have a well-defined plane of observation B in which the images that are viewed by it are located.

Two small marks are made on the outer surfaces O, O' of the extreme lenses of the system. Let S, S' stand for the images of these marks O, O' when viewed through the lenses. The system is mounted on a graduated slide and arranged so that light from a distant object is brought to a focus at B. We know then that the second principal focus F_2 is at B. The system is then moved up towards the microscope so that the mark O' becomes visible, *i.e.* O' is at B. The difference of the readings gives the distance F_2O'. The system is moved nearer so that the mark O is visible through the microscope, *i.e.* S is at B. The difference of the last two readings gives the distance $O'S$. By reversing the combination the distances F_1O and OS' can be determined in like manner. Then we obtain two determinations of the focal length from

$$-f^2 = F_1O \,.\, F_2S = F_1S' \,.\, F_2O',$$

and having found the principal foci and the focal length, the positions of the unit points are determined.

114. Any number of media separated by coaxial spherical surfaces.

Let $\mu_0, \mu_1, \ldots \mu_n$ be the refractive indices, $r_1, r_2, \ldots r_n$ the radii of the successive surfaces, and $T_1, T_2, \ldots T_{n-1}$ the distances between them. Let U, V_1 denote the distances from the first surface of a point on the axis and its image formed by refraction at the first surface, and let $V_2, V_3, \ldots V_{n-1}, V$ denote the distances of the succeeding images from the corresponding surfaces, with the usual convention about signs.

Then we have

$$\frac{\mu_1}{V_1} - \frac{\mu_0}{U} = \frac{\mu_1 - \mu_0}{r_1},$$

$$\frac{\mu_2}{V_2} - \frac{\mu_1}{V_1 + T_1} = \frac{\mu_2 - \mu_1}{r_2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\mu_n}{V} - \frac{\mu_{n-1}}{V_{n-1} + T_{n-1}} = \frac{\mu_n - \mu_{n-1}}{r_n},$$

* A. Cornu, *Journal de Physique*, t. VI. p. 276. 1877.

or if $k_1, k_2, \ldots k_n$ denote the powers of the surfaces and $U = \mu_0 u$, $V_1 = \mu_1 v_1, \ldots V = \mu_n v$, $T_1 = \mu_1 t_1$, $T_2 = \mu_2 t_2$, $\ldots T_{n-1} = \mu_{n-1} t_{n-1}$, these equations take the form

$$\frac{1}{v_1} - \frac{1}{u} = k_1,$$

$$\frac{1}{v_2} - \frac{1}{v_1 + t_1} = k_2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1}{v} - \frac{1}{v_{n-1} + t_{n-1}} = k_n.$$

As in Art. 92, this leads to a result

$$v = \frac{Cu + D}{Au + B},$$

where C/A and D/B denote the last but one and last but two convergents to the continued fraction

$$\frac{1}{k_n +} \frac{1}{t_{n-1} +} \frac{1}{k_{n-1} + t_{n-2} +} \ldots \frac{1}{+ t_1 +} \frac{1}{k_1 +} \frac{1}{u},$$

and $BC - AD = 1$.

115. Linear magnification.

As in Art. 93 it may be shewn that the linear magnification m

$$= \frac{1}{Au + B} = C - Av.$$

116. As the relations obtained in the last two articles are of exactly the same form as those obtained for three media separated by two spherical surfaces, it is not necessary to carry this investigation further. All the results and formulae deduced for that case are true in the general case when there are any number of media.

117. Combination of two instruments or sets of lenses on the same axis.

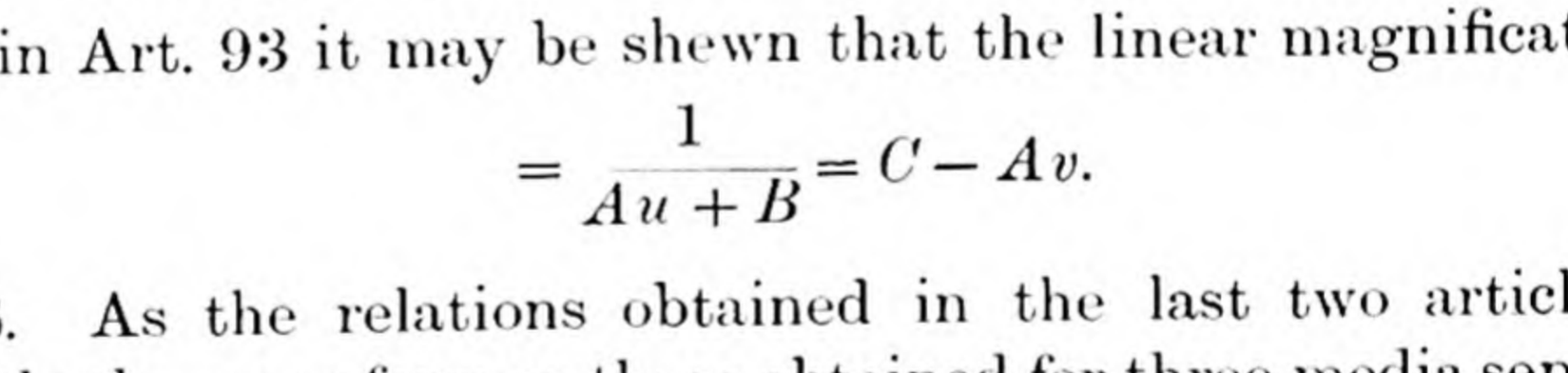

Fig. 78.

Let the instruments be represented by their cardinal points F_1, F_2, H, H' and f_1, f_2, h, h'. Let F, f be their focal lengths and

c the distance f_1F_2. Let X_1, X_2, and Y, Y' denote the principal foci and unit points of the combination.

Rays which emerge parallel to the axis must be directed through f_1 before they pass through the second instrument, therefore the first principal focus X_1 of the combination must be the conjugate of f_1 with regard to the first instrument.

Hence $$F_1X_1 . F_2f_1 = -F^2 \text{ or } F_1X_1 = F^2/c.$$

Similarly the second principal focus X_2 of the combination is the conjugate of F_2 with regard to the second instrument, so that

$$f_2X_2 . f_1F_2 = -f^2 \text{ or } f_2X_2 = -f^2/c.$$

Again let Q denote an object to the right of F_1 of which the first instrument forms an image Q_1 between f_1 and F_2, and the second instrument forms an image Q' to the left of f_2. The total linear magnification M

$$= \frac{F_2Q_1}{F_2H'} \cdot \frac{f_1h}{f_1Q_1};$$

and Q, Q' will coincide with the unit points Y, Y' of the combination, provided $M = 1$, that is, if

$$\frac{Q_1F_2}{F} = \frac{f_1Q_1}{f} = \frac{c}{F+f}.$$

Therefore $$F_1Y = -F^2/F_2Q_1 = F(F+f)/c,$$

and $$f_2Y' = -f^2/f_1Q_1 = -f(F+f)/c.$$

Again, the focal length of the combination

$$= X_1Y = F_1Y - F_1X_1 = Ff/c.$$

Thus the cardinal points of the combination are completely determined.

EXAMPLES.

1. The surfaces of a concavo-convex piece of glass of thickness $\frac{1}{4}$ of an inch and index of refraction equal to $\frac{3}{2}$ are of equal radius 6 inches, and a concave mirror is formed by silvering the convex surface. Prove that the silvered glass mirror is optically equivalent to a simple concave mirror of radius approximately equal to 5·797 inches, placed with its centre of curvature at a distance from that of the front surface of the glass approximately equal to 0·367 inches.

A source of light being placed on the axis at a distance 8 inches in front of the front surface, mark on a figure drawn to scale the images formed (i) by refraction at the front surface, (ii) by such refraction followed by reflection at the silvered surface, (iii) by the complete silvered glass mirror.

(M. T. ii. 1912.)

2. A thin double-convex glass lens, whose surfaces are portions of spheres of 6 inches radius, is held in the surface of water so that the lower face is submerged. Find the position and the size of the image of the sun formed by the lens, assuming that the sun is vertically above the lens, and that its diameter subtends at the earth an angle whose circular measure is 0·009. [Take the index of glass to be $\frac{3}{2}$, and that of water to be $\frac{4}{3}$.]

(St John's Coll. 1907.)

3. A glass sphere of radius r and refractive index μ is half immersed in fluid of refractive index μ', and a small object at a depth z vertically below the centre is observed by an eye vertically above the centre. Prove that, if

$$\frac{z}{r} = \frac{\mu(\mu'-1)}{2\mu' - \mu(\mu'+1)},$$

the image will coincide with the object. (M. T. 1909.)

4. A sphere of glass ($\mu = \frac{3}{2}$) of diameter 2 inches is bisected by a horizontal plane and the upper hemisphere is raised a distance $\frac{1}{3}$ inch above the lower; prove that the two hemispheres placed thus will form a virtual and enlarged image of any small object which is vertically below the lowest point of the lower hemisphere and less than $\frac{2}{7}$ inch from it. (M. T. 1896.)

5. Two hemispheres of radius r are placed with their curved surfaces towards one another on the same axis, the shortest distance between the curved surfaces being t, and the refractive indices μ, μ'; shew that the linear magnification of a small object on the axis will be independent of its distance provided

$$(\mu-1)(\mu'-1)\,t = (\mu+\mu'-2)\,r. \qquad \text{(Coll. Exam. 1898.)}$$

6. A glass sphere of index μ is cut in half and the two hemispheres are separated, the line joining their centres being perpendicular to their plane faces, which oppose each other. Prove that if u, v be the distances of two conjugate foci from the first and second curved surfaces respectively, then the corresponding magnification is $(c-v)/(c-u)$, the radius of the sphere being $(\mu-1)\,c$. Find also the magnification when $u = c$. (St John's Coll. 1896.)

7. Two equal glass hemispheres, of radius r and refractive index $\frac{3}{2}$, are placed so that the vertex of the first touches the centre of the plane base of the second, and the arrangement is symmetrical about the line joining the centres. A small object is placed on this line at a distance r beyond the second hemisphere. Prove that a real inverted image is formed by refraction through the system, and that its size is six times that of the object. Find also the position of the image. (M. T. 1905.)

8. If the radius of a sphere is one inch and a luminous point is 8 inches from the centre of the sphere, prove that the position of the geometrical focus will be unaltered if the sphere be cut in two by a diametral plane perpendicular to the axis of the pencil, and the remoter half be moved back a distance $5\frac{5}{12}$ inches; the refractive index of the material of which the sphere consists is $\frac{4}{3}$. (Coll. Exam. 1902.)

9. If a system consists of two thin divergent lenses, each in contact with a sphere whose radius a is equal to the focal length of either lens, prove that the equivalent lens is a divergent lens of focal length $\frac{a\mu}{4(2-\mu)}$ placed at a distance $a/2$ from the centre of the sphere, on the side of the incident light; μ being the refractive index of the sphere. (M. T. 1904.)

10. Shew that the formula $\kappa_1+\kappa_2+a\kappa_1\kappa_2$ will hold for the equivalent lens of two thick lenses, if a denote the distance from the second plane of unit magnification of the first lens to the first plane of unit magnification of the second lens, κ_1 and κ_2 being the powers of the lenses. (Trinity Coll. 1891.)

11. If there are two thick lenses on the same axis whose focal lengths are ϕ, ϕ', and κ be the distance between the second principal point of the first and the first principal point of the second, shew that the distances of the focal points of the combination from the first principal point of the first and the second principal point of the second are in the ratio

$$1+\frac{\kappa}{\phi'} : 1+\frac{\kappa}{\phi}.$$ (Coll. Exam. 1891.)

12. An optical system consists of two equal hemispherical lenses whose curved surfaces are in contact and plane faces parallel, the refractive index and radius of the curved surface of each hemisphere being μ and a. Shew that, for a pencil refracted directly through the system, the magnification m is given by the equation

$$1/m=1-2\kappa a/\mu-2\kappa u,$$

where $\kappa=(\mu-1)/a$, and u is the distance of the object from the nearest plane face. (Coll. Exam. 1912.)

13. Two equal glass spheres of refractive index μ are in contact and a ray nearly parallel to the common diameter is directed towards the point of contact of the spheres. Prove that the ratio of the inclinations of the ray to the common diameter after and before refraction is $1-8\left(1-\frac{1}{\mu}\right)^2$.

(Coll. Exam. 1901.)

14. Two thin convex lenses of refractive index $\frac{3}{2}$ and equal power $1/2a$ are placed on the same axis at a distance $2a$ apart. A sphere of the same material and of radius a is placed between the lenses so that its centre lies on the axis of the lenses. Shew that the minimum distance between a real object and a real image of it is $4a$. (Coll. Exam. 1906.)

15. A plano-convex glass lens of index μ and thickness t, having a spherical surface of radius r, is held so that its plane face is downwards and is just in the surface of a vessel of water of index μ'. Find the principal focal lengths of the instrument so formed, and the positions of the planes of unit magnification. Shew that, if the sun be in the zenith and have an angular diameter of δ radians, the linear diameter of its image formed in the water is $r\delta/(\mu-1)$. (M. T. ii. 1912.)

16. Prove that when a mirror (plane or curved) is placed behind a symmetrical instrument, the reflected image of an object placed in front of the instrument is exactly the same as that produced by a single mirror, and determine the positions of the centre and vertex of this equivalent mirror.
(St John's Coll. 1911.)

17. A mirror has its front face flat, and its silvered back is part of a sphere of radius r; the thinnest part of the glass is in the centre and its refractive index is μ. Shew that the radius of the equivalent mirror is r/μ, and find its vertex. (St John's Coll. 1911.)

18. Prove that the focal length f of a thick convex lens may be determined thus. Taking a fixed point of light and a fixed screen, measure the distance c between the two positions of the lens in which it gives an image of the light focused on the screen. Move the screen an additional distance h from the light, and performing a similar operation, measure the corresponding distance c'. Then f will equal

$$\{-s(s-h)(s-c)(s-c')\}^{\frac{1}{2}}/h,$$

where $2s = h + c + c'$. (Coll. Exam. 1891.)

19. If n equal spheres of radius a and refractive index $\sec^2\theta$ are placed in contact with their centres on the same straight line, shew that the points of unit magnification are at distance $a(1-\tan n\theta \cot\theta)$ from the centres of the end spheres. (M. T. 1903.)

20. Light is refracted through a series of n concentric spherical shells of glass of refractive index μ. If the radii of all the surfaces form a harmonic progression, R, r being the greatest and least, prove that the equivalent lens of the system is f where

$$\frac{1}{f} = \frac{2(\mu-1)}{\mu}\,\frac{n}{2n-1}\left(\frac{1}{r}-\frac{1}{R}\right). \qquad \text{(Coll. Exam. 1896.)}$$

21. Explain what is meant by the single refracting surface *equivalent* to a system of refracting surfaces disposed along a common axis of symmetry. Find the radius of curvature of the equivalent surface for media μ_1, μ_2, μ_3, μ_4 separated by surfaces of radii r_1, r_2, r_3, at distances t_1, t_2 apart, the concavities being all towards the medium μ_4. What takes the place of the equivalent surface if $\mu_1 = \mu_4$? (St John's Coll. 1903.)

22. Given two optical systems on an axis and the positions of the pairs of conjugate points for which the linear magnification is $+1$ and -1 respectively for each system separately, construct geometrically the positions of the corresponding pairs of conjugate points for the combination of the two systems. (Coll. Exam. 1904.)

23. Two symmetrical optical instruments are placed so as to be coaxial. An incident ray parallel to the common axis cuts the first focal and first unit planes of one of the instruments in two points. Give a geometric construction for the images of these points formed by the combination, having given the foci and unit planes of each instrument. Hence shew how to construct the foci and unit planes of the combination. (M. T. ii. 1913.)

CHAPTER VIII

DISPERSION AND ACHROMATISM

118. It can be shewn by experiment that ordinary sunlight is composite and consists of the superposition or blending of lights of a variety of colours and different refrangibilities.

We shall describe a few of the historic experiments on this subject as performed by Newton. For a fuller account of his investigations the reader is referred to his own writings*.

119. First Experiment.

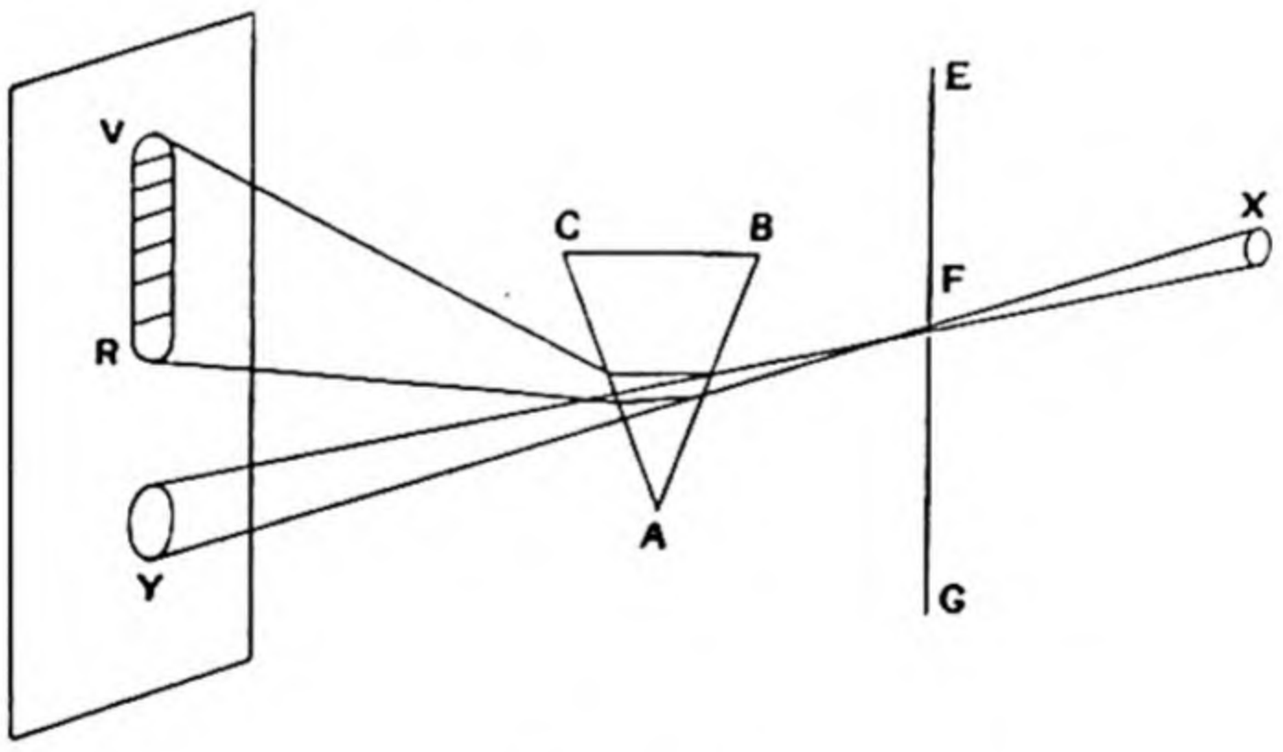

Fig. 79.

A beam of sunlight enters a darkened room through a small opening F in a shutter EFG and produces a bright circular spot Y on the opposite wall. If a prism ABC is inserted in the beam with its edge horizontal downwards and parallel to the wall, a vertical elongated strip VR composed of horizontal bands of colour of the same width as the spot Y is formed on the wall. The colours are violet, indigo, blue, green, yellow, orange, red in the order named, the violet being highest. This band of colour is called a **spectrum**.

* *Loc. cit.* p. 2.

120. Second Experiment.

This experiment is to prove what the former suggested, namely, that the coloured constituents of sunlight are of different refrangibilities. A second prism DH is interposed in the beam,

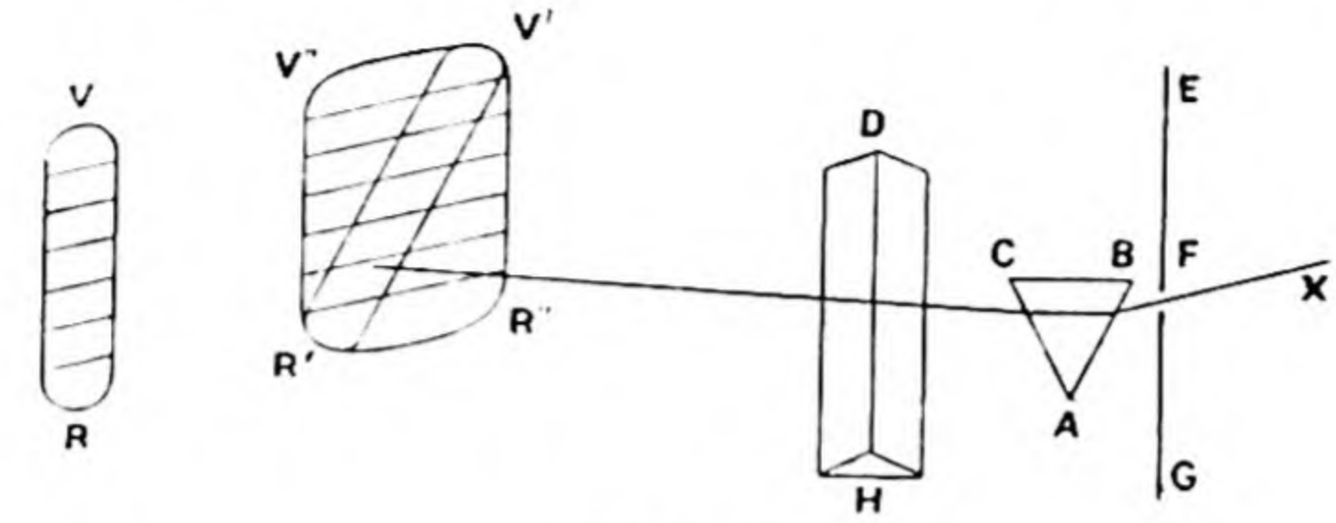

Fig. 80.

with its axis vertical. The vertical strip VR is displaced and becomes an oblique strip $V'R'$. If the different coloured lights possessed the same property as white sunlight the violet strip V ought to be drawn out by the second prism into an elongated horizontal strip $V'V''$ and similarly the red strip R into $R'R''$, and the intermediate coloured strips in the same way, but this does not happen, and the formation of the oblique strip $V'R'$ shews that the coloured lights are of different refrangibility, the refrangibility increasing continuously from the red to the violet.

121. Third Experiment.

The same property is demonstrated again by the following experiment.

Two boards DE, de with holes G, g are placed in the beam of light after it passes through the prism ABC, so that the central part of the beam passes through holes at G and g in the boards. The light that gets through these apertures then traverses a prism abc and produces a bright spot at M on the screen MN.

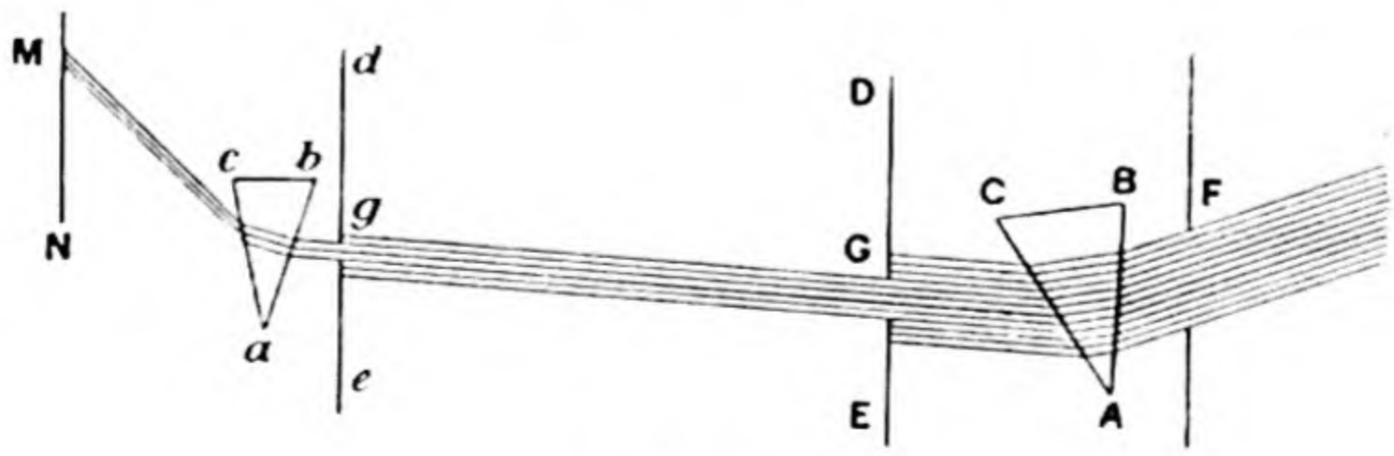

Fig. 81.

By keeping the boards and the prism abc fixed the angle of incidence on the prism abc is fixed, but it is found that when the

prism ABC is rotated the spot at M changes position, and shews that the rays which are most refracted by the first prism are also most refracted by the second.

122. Fourth Experiment.

Another experiment in verification of the same phenomenon is based on the property of total reflection. The critical angle $\sin^{-1}(1/\mu)$ is different for rays of different refrangibility and the violet constituent will be totally reflected at an angle of incidence in glass for which the red constituent will emerge.

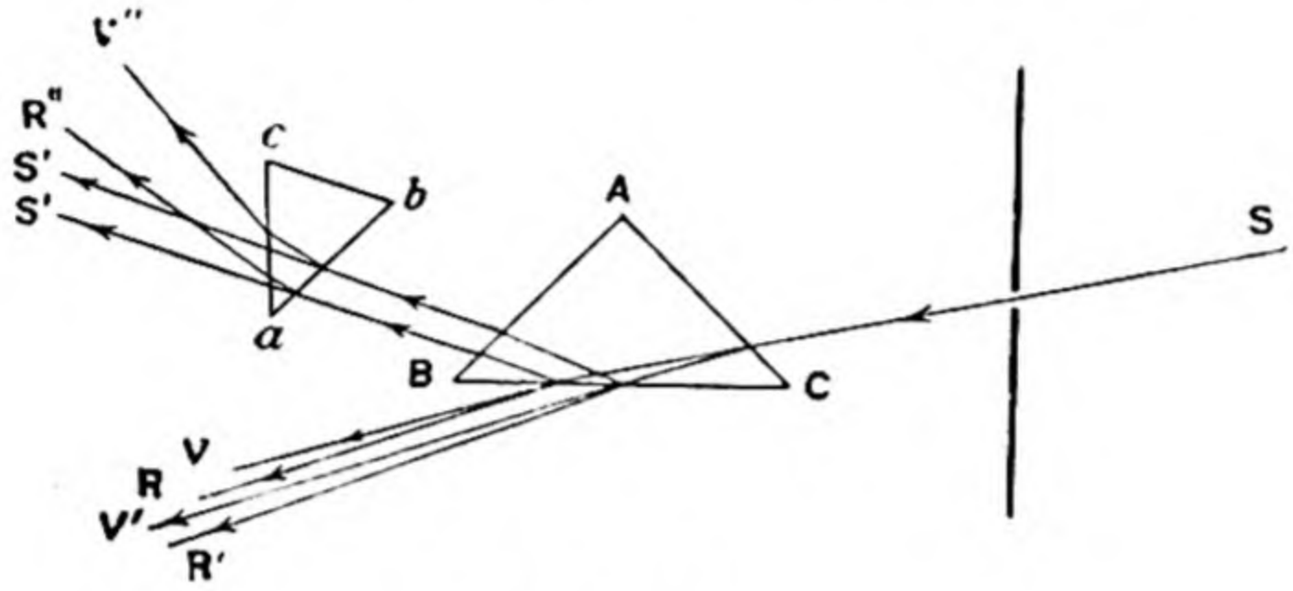

Fig. 82.

Thus if a beam incident from S on the face AC of an isosceles prism ABC emerges from the base BC a spectrum VR' is formed. If the prism is slowly rotated so as to increase the angle of incidence at BC, it is found that a point is reached at which the violet rays disappear from the spectrum, being reflected internally and emerging in direction S'. As the prism is rotated further in the same direction the other colours in turn disappear from the spectrum VR', and as the rays emergent from AB make the same angle as the rays incident on AC the successive colours will be superposed and produce white light in direction S'. If however another prism abc be inserted in the beam the colours are separated and as ABC is rotated the different colours will appear in succession forming a spectrum $V''R''$.

123. The spectrum described in the foregoing experiments is not a pure spectrum as it consists of overlapping images of the sun's disc. In order to separate the colours an achromatic lens L may be placed in front of the prism (see Fig. 83). This forms an image I of the hole or slit F through which the beam of light is admitted, and if the prism ABC is placed so that the mean ray passes through it with minimum deviation a series of distinct coloured images $V \ldots R$ of I is formed.

For more precise means of obtaining a pure spectrum the reader is referred to text-books on Physics or the Spectroscope.

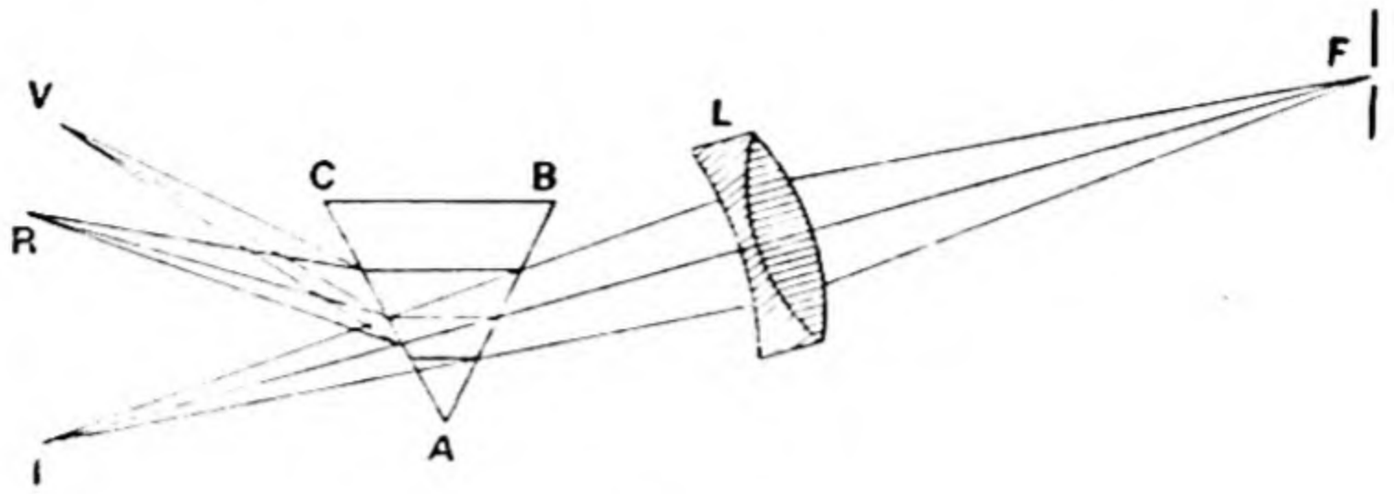

Fig. 83.

124. When a pure solar spectrum is obtained it is found to be traversed by certain dark lines. The explanation of the dark lines is found in the facts that there are three kinds of spectra, viz. (1) solar and stellar spectra, which are continuous save for the interruption of definite dark lines; (2) spectra due to incandescent solids and liquids, which are continuous; (3) spectra due to incandescent gases which consist of a few bright lines only.

125. It is found that some substances emit certain rays to the exclusion of others and that every such substance absorbs the same kind of rays as it emits. Thus if a bright source of light be giving a continuous spectrum, and an incandescent gas be interposed between the source of light and the point of observation, if the gas be not so bright as the source it will absorb brighter rays than it emits and the result will be some dark lines in the otherwise continuous spectrum. If the bright source of light be removed and the spectrum due to the incandescent gas alone be examined it will be found to consist of just those lines which were formerly dark. We conclude that the sun's light arises from an inner nucleus which would give a continuous spectrum but that this nucleus is surrounded by gases which absorb certain rays and so produce the dark lines in the spectrum. Spectrum analysis is now of the utmost importance in the study of the composition of the heavenly bodies.

126. Dispersive power.

When a ray of white light is split up or dispersed by a prism the difference of the deviations of two colours is called the **dispersion** of those colours.

To measure the **dispersive power** of a substance let D be the deviation produced in some standard ray selected out of the brightest part of the spectrum produced by a prism of *small* refracting angle i, and let μ be the coefficient of refraction of that ray.

Then we have $D = (\mu - 1)\, i$. (Art. 42.)

Now if D_1, D_2 denote the deviations of coloured rays whose indices of refraction are μ_1, μ_2, we have

$$D_1 = (\mu_1 - 1)\, i, \quad \text{and} \quad D_2 = (\mu_2 - 1)\, i.$$

Hence
$$\frac{D_1 - D_2}{D} = \frac{\mu_1 - \mu_2}{\mu - 1},$$

and this expression is called the *dispersive power* of the substance for the two selected coloured rays, and is denoted by ϖ.

It is to be noted that the value of ϖ does not depend on the form of the prism but that it does depend on the *three* rays with regard to which comparison is being made. The difference $\mu_1 - \mu_2$ may be denoted by $\delta\mu$, where δ denotes the increment obtained by changing μ from μ_2 to μ_1, and then $\varpi = \delta\mu/(\mu - 1)$.

127. Irrationality of dispersion.

This means that the ratios of the dispersive powers of two different media for two pairs of coloured rays are not equal.

For example if μ, $\mu + \delta\mu$ are the refractive indices for two rays A and B, and ν, $\nu + \delta\nu$ are the refractive indices for two rays C and D, in crown glass, and μ', $\mu' + \delta\mu'$, ν', $\nu' + \delta\nu'$ denote the same quantities for flint glass, the ratio of the dispersive powers for crown glass is

$$\frac{\delta\mu}{\mu - 1} : \frac{\delta\nu}{\nu - 1},$$

and for flint glass it is

$$\frac{\delta\mu'}{\mu' - 1} : \frac{\delta\nu'}{\nu' - 1},$$

and these two ratios are not equal.

This property of irrationality of dispersion was overlooked by Newton* who concluded that it was impossible to correct dispersion without destroying deviation of the rays at the same time, and he therefore designed and made a reflecting telescope as he was convinced that "the improvement of Telescopes of given lengths by Refractions is desperate."

128. Achromatism.

When an object is viewed through an optical instrument, owing to the different refrangibilities of lights of different colour the various coloured images that are formed are generally not of the same size and not in the same position, so that the image

* *Optics*, Prop. VII.

appears with a fringe of colour. This defect of the image is called **chromatic aberration**, a combination of lenses or prisms free from this defect is said to be **achromatic.**

129. Achromatic combination of prisms.

Let there be a number of prisms of small angles $i, i', i''\ldots$ placed with their edges parallel. Let their refractive indices for a standard ray be $\mu, \mu', \mu'' \ldots$, and for a ray of another colour $\mu + \delta\mu$, $\mu' + \delta\mu'$, $\mu'' + \delta\mu'' \ldots$.

The total deviation produced in the standard ray is

$$D = (\mu - 1)\, i + (\mu' - 1)\, i' + (\mu'' - 1)\, i'' + \ldots$$

and the dispersion of the two rays is

$$\delta D = i\, \delta\mu + i'\, \delta\mu' + i''\, \delta\mu'' + \ldots.$$

The condition that there should be no dispersion is therefore

$$i\, \delta\mu + i'\, \delta\mu' + i''\, \delta\mu'' + \ldots = 0.$$

If the prisms are all of the same kind of glass, we have

$$\mu = \mu' = \mu'' = \ldots$$

and

$$D = (\mu - 1)\,(i + i' + i'' + \ldots).$$

Also the condition for no dispersion is

$$\delta\mu\,(i + i' + i'' + \ldots) = 0,$$

or

$$\frac{\delta\mu}{\mu - 1} D = 0.$$

Therefore $D = 0$, that is, there is no deviation. Hence with prisms of the same kind of glass it is impossible to deflect a ray without producing dispersion.

130. By suitable choice of the angles of two prisms of different materials it is always possible to combine two rays of different colour. For the condition for no dispersion, as far as the two selected rays are concerned, is

$$i\, \delta\mu + i'\, \delta\mu' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

But it is not possible with the same two prisms to combine a third ray of a different colour. For if the refractive indices for this ray in the two prisms be

$$\mu + \partial\mu \text{ and } \mu' + \partial\mu',$$

the condition that there shall be no separation between this ray and the first is

$$i\partial\mu + i'\partial\mu' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and, if (1) and (2) hold simultaneously, we have

$$\frac{\delta\mu}{\mu-1} : \frac{\delta\mu'}{\mu'-1} = \frac{\partial\mu}{\mu-1} : \frac{\partial\mu'}{\mu'-1},$$

which is impossible because of irrationality of dispersion. Hence with two prisms only two coloured rays can be combined. With three prisms three colours can be combined and so on.

131. Dispersion produced by a prism of finite angle.

With the usual notation μ denotes the refractive index of a standard ray and the angles ϕ, ϕ', ψ, ψ' in this figure relate to this ray. Let $\phi + \delta\phi$, $\phi' + \delta\phi'$, $\psi + \delta\psi$, $\psi' + \delta\psi'$ denote corresponding angles for a ray of refractive index $\mu + \delta\mu$.

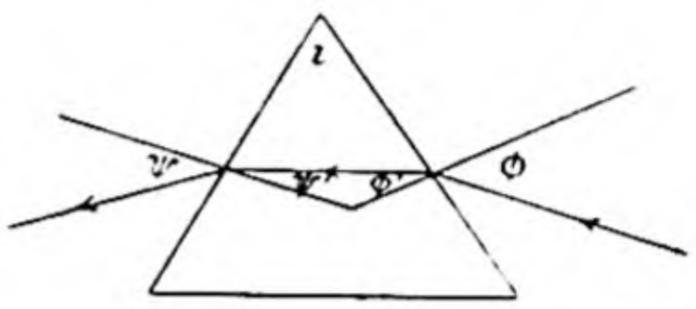

Fig. 84.

We have

$$\sin\phi = \mu\sin\phi', \quad \sin\psi = \mu\sin\psi'$$

and

$$\phi' + \psi' = i.$$

Hence

$$\cos\phi\delta\phi = \delta\mu\sin\phi' + \mu\cos\phi'\,\delta\phi',$$

$$\cos\psi\delta\psi = \delta\mu\sin\psi' + \mu\cos\psi'\,\delta\psi',$$

and

$$\delta\phi' + \delta\psi' = 0.$$

Therefore

$$\cos\phi\cos\psi'\delta\phi + \cos\phi'\cos\psi\delta\psi = \sin(\phi' + \psi')\,\delta\mu$$
$$= \sin i\,.\,\delta\mu,$$

giving the relation between the dispersions at incidence and emergence.

If the rays are combined at incidence $\delta\phi = 0$

and

$$\delta\psi = \frac{\sin i\,\delta\mu}{\cos\phi'\cos\psi}.$$

132. Achromatism of lenses.

An image formed by a lens is affected by dispersion in two ways—(1) the different coloured images are distributed at different distances along the axis; (2) their magnitudes are different. It is not always possible to correct both defects even as regards the combination of two colours. When only one defect is corrected the achromatism is said to be *partial*. When each point and its image lie on a straight line passing through a fixed point as for example the centre of a thin lens, then if all the coloured images are brought

into one plane they will all have the same size and the correction of the first fault will also correct the second.

133. Effect of dispersion on Focal Length.

The focal length of a lens is different for different colours. If f be the focal length of a thin lens for a standard ray of refractive index μ and r, s are the radii of the surfaces, we have

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right).$$

Hence $$\delta\left(\frac{1}{f}\right) = \delta\mu\left(\frac{1}{r} - \frac{1}{s}\right) = \frac{\delta\mu}{\mu - 1}\frac{1}{f} = \frac{\varpi}{f};$$

or if k denote the power of the lens

$$\delta k = \varpi k.$$

134. Achromatism of two thin lenses.

First when the lenses are in contact, the correction for size of image will also ensure that the images are formed in the same position and vice versa.

If k, k' are the powers of the lenses and ϖ, ϖ' the dispersive powers of the two kinds of glass for the colours which it is proposed to combine, we get from equations (1) and (2) of Art. 79,

$$\frac{1}{v} - \frac{1}{u} = k + k'.$$

The condition for achromatism is that v should be the same for the two colours, hence we must have

$$\delta(k + k') = 0$$

or $$\varpi k + \varpi' k' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

We observe that this condition is independent of the position of the object, and that if the power A of the combination is given the required powers of the two lenses that will combine the two colours are given by

$$k + k' = A$$
$$\varpi k + \varpi' k' = 0.$$

Also, that if the lenses are of the same kind of glass so that $\varpi' = \varpi$ the condition for achromatism becomes

$$k + k' = 0,$$

shewing that it is impossible to make an achromatic combination of two lenses in contact possessed of any power if the lenses are not of different kinds of glass.

135. Object-glasses.

The condition (1) shews that of the two lenses one must be convergent and the other divergent. For an object-glass we require convergence, and, as flint-glass produces nearly twice as much dispersion as crown-glass, a suitable combination consists of a convergent lens of crown-glass and a less powerful divergent lens of flint-glass. The latter corrects the dispersion produced by the former but leaves the combination as a whole convergent.

136. Secondly, when the lenses are separated by a distance a. We shall see that it is impossible in this case to effect the corrections for both size and position of image. We shall find the condition for achromatism as regards the magnitude of two coloured images. By Helmholtz's formula it will follow that the two selected coloured components of a ray proceeding from the object emerge parallel to one another.

If m be the linear magnification, we have, from Art. 81 (5)

$$\frac{1}{m} = u\,(akk' + k + k') + ak' + 1.$$

Hence $$\delta\left(\frac{1}{m}\right) = u\,\{akk'\,(\varpi + \varpi') + \varpi k + \varpi' k'\} + a\varpi' k',$$

so that the condition for the equal magnitude of the two coloured images is

$$u\,\{akk'\,(\varpi + \varpi') + \varpi k + \varpi' k'\} + a\varpi' k' = 0 \ldots\ldots\ldots(1).$$

We observe that this condition depends on the position of the object, and can only be independent of u if

$$a\varpi' k' = 0;$$

that is, if $a = 0$, when the condition reduces to that of Art. 134; or, if $\varpi' = 0$, and this is not possible, for there is no substance which will produce deviation without dispersion.

It is always possible to choose a, k, k' so as to satisfy (1) *for a prescribed value of* u, but this is a *particular* rather than a *general achromatism*, and it is partial in that it only secures equality of size of two images and not coincident positions.

In order to secure this latter condition *as well*, we must have $\delta v = 0$ as well as $\delta\left(\frac{1}{m}\right) = 0$, and in the notation of Arts. 79—81

$\frac{1}{m} = \frac{u\,(v_1 + a)}{v_1 v}$ so that we must have $\delta v_1 = 0$ and this would mean that the first lens was achromatic of itself, which cannot be.

137. Eye-pieces.

For an eye-piece consisting of two thin lenses made of the same kind of glass, condition (1) of the last article reduces to

$$u(k + k' + 2akk') + ak' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

This condition is independent of σ and therefore when it is satisfied all the coloured images are of the same size. It depends however on the value of u, that is, on the position of the object. In a telescope in normal adjustment the object-glass forms an image in the common focal plane of the object-glass and eye-piece (Art. 177). But the eye-piece has different focal planes for different colours, so that, though the object-glass may be achromatic, rays of the same colour will not emerge from the eye-piece as parallel rays unless the image formed by the object-glass is in the focal plane of the eye-piece for that particular colour.

If we make the above condition hold for rays parallel to the axis, *i.e.* for $u = \infty$, it reduces to

$$k + k' + 2akk' = 0,$$

or

$$a = -\tfrac{1}{2}(f + f') \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

This rule, that the distance between the lenses is the arithmetic mean of their focal lengths, is satisfied by Huyghens' eye-piece (see Art. 189).

If we want the eye-piece to be achromatic for a ray through the centre of the object-glass, the condition (1) becomes

$$k + k' + 2akk' + \frac{a}{u}k' = 0,$$

where u is now the distance between eye-piece and object-glass. In most telescopes the fraction a/u is small and the condition reduces to (2) as before.

138. General system of lenses.

The position and magnitude of the image depend on the positions of the principal foci and the focal length. Hence for achromatism the positions of the principal foci and the power of the system ought to be independent of colour. These depend on the constants in the formula

$$v = \frac{Cu + D}{Au + B},$$

where $BC - AD = 1$, so that there are only three independent constants. Hence three quantities, say A, B, C, are to be independent of colour. We must therefore have three variables at our disposal, so, if the powers of the lenses are given, we must have at least four lenses, and adjust their distances so as to satisfy the requisite conditions. The calculations however prove to be too complex for practical use.

If α_0, α are the inclinations to the axis of an incident and emergent ray, and u be the distance from the first lens at which the incident ray crosses the axis, we have by Helmholtz's Theorem

$$\frac{\alpha}{\alpha_0} = \frac{1}{m} = Au + B.$$

Therefore $$\delta\alpha = \alpha_0 (u\,\delta A + \delta B).$$

In order that the emergent coloured rays shall be parallel we require $\delta\alpha$ to be zero, hence we must have

$$u\,\delta A + \delta B = 0.$$

As this depends on the position of the object it is usual to make $\delta A = 0$ the first condition to be satisfied, or in words, *that the power shall be independent of colour*. Then we have $\delta\alpha = \alpha_0 \delta B$, as the amount of the remaining dispersion, and owing to the smallness of α_0 this term may be small enough to neglect, but if not we must endeavour to satisfy $\delta B = 0$ also.

If further we require the coloured images to be in the same position as well as of the same size we must also satisfy $\delta C = 0$.

EXAMPLES.

1. A prism of 60° angle is constructed of glass of which the index is 1·5 for a red ray and 1·56 for a violet ray. Find the angular separation produced by it on these rays, when a narrow parallel beam of white light is incident at minimum deviation. At what distance from the prism would the spectrum produced be 12 inches long from red to violet? (St John's Coll. 1903.)

2. A beam of light is incident on a glass prism of angle 60°, making an angle of 75° with the normal to the first surface. Calculate the deviation produced in the colours of the beam for which $\mu = 1{\cdot}54$ and in those for which $\mu = 1{\cdot}56$; and verify that the angle between the corresponding emergent rays is about 1° 30′. (St John's Coll. 1909.)

3. An equilateral prism is held close to a small hole in the shutter of a dark room with its edges perpendicular to the sun's rays. Describe the appearance of the patch of light seen on a white screen held perpendicularly

to the refracted mean ray at a distance of 15 feet from the prism, and calculate its approximate dimensions when the ray passes through the prism with minimum deviation. Take the indices of refraction for the violet and red rays to be 1·533 and 1·513, and the sun's angular diameter to be 32′.

A patch of light is formed on the ceiling of a room by light from the sun which has been internally reflected at the horizontal base of an equilateral prism. Determine the position and describe the colouring of the patch.

(M. T. ii. 1913.)

4. Two rays of different colours are incident at the same point in the same direction on a plane face of a prism of angle i; one passes through with minimum deviation D, and the other with a slightly greater deviation $D+\theta$; prove that after emergence they cut a screen at right angles to the first ray at points whose distance apart is approximately equal to

$$\left(r + a \tan^2 \frac{i}{2} \cot \frac{D+i}{2} \cos \frac{D+i}{2}\right)\theta,$$

where r is the length of the path of the first between the prism and the screen, and a is the distance of the point of incidence from the edge.

(St John's Coll. 1897.)

5. A beam of light enters a wide-angled prism, whose cross section is a triangle ABC, by the face AB, is reflected at the base, and emerges at the other face AC; shew that, in case AB is greater than AC, and AD is inflected on the base so as to be equal to AC, the dispersion of colours in the emergent beam is the same as if it had been merely refracted through the prism DAB of smaller angle. (St John's Coll. 1900.)

6. Two lenses of the same kind of glass, one convex the other concave, and each of 3 inches focal length, are placed 8 inches apart, and a small white object is placed 12 inches beyond the convex lens. Shew that the positions of the various coloured images are the same. (St John's Coll. 1905.)

7. An objective lens consists of a convergent lens (α) and a divergent lens (β) having a common surface of radius 60 cms. The radius of the other surface of (α) is 56 cms. and the other surface of (β) is plane. The mean refractive indices of the two kinds of glass are (α) 1·5760 and (β) 1·6202 and the dispersive powers, measured by the ratio of the difference of the indices for two different rays to the mean index minus unity, are (α) ·015 and (β) ·027. Determine whether the combination is over- or under-corrected, and shew that the difference of focal lengths for these two standard rays is approximately 2 mm. (M. T. ii. 1910.)

8. A divergent and a convergent lens of the same material and of mean focal lengths f and $-4f$ are placed on the same axis at a distance a. Prove that a small object at a distance $af/(f-a)$ outside the former lens has an image whose position is the same for rays of different colours.

(Coll. Exam. 1906.)

9. A thin convex lens of focal length a is placed midway between two thin concave lenses each of focal length $\frac{1}{2}a$, which are at distance $2a$ apart. All three lenses are of the same material. Shew that a pencil of rays, which is converging towards the centre of the middle lens, will after passing through the system diverge from this point achromatically, squares and cubes of the dispersive power of the material being neglected. (M. T. 1896.)

10. A plano-concave lens and a plano-convex lens having their curved surfaces of the same radius r are placed in contact on the same axis with their plane faces outwards. Shew that if μ_1, μ_2 be the indices of refraction of the two lenses the positions of the focal points of the combination will be the same for two colours if

$$a\delta\left(\frac{1}{\mu_1}\right)=b\delta\left(\frac{1}{\mu_2}\right)=r\delta\left(\frac{1}{\mu_2-\mu_1}\right),$$

where a, b are the respective thicknesses of the lenses measured along their axis. (Coll. Exam. 1908.)

11. The dispersive power of the substance of a lens for two given colours is ϖ, and its focal length for one of the colours is f. A pencil of light consisting of the two colours is emitted from a point on the axis, and after refraction its focus for one of the colours is at distance v from the lens. Prove that the smallest ring through which the refracted pencil of both colours will pass is of radius $\frac{1}{2}\varpi y v/f$, where y is the radius of aperture of the lens. (Trinity Coll. 1904.)

12. Two lenses of focal lengths f_1, f_2 are made of the same kind of glass and an object placed at a distance u from f_1 is viewed by an eye placed at a distance x from f_2; shew that the different coloured images will have the same angular diameter if

$$\{x(2a+f_1+f_2)+af_2\}u+axf_1=0. \quad \text{(Coll. Exam. 1896.)}$$

13. A system, composed of two thin lenses of the same glass and of powers κ_1 and κ_2 at a finite distance apart, satisfies the conditions of achromatism of an eye-piece for parallel rays. An object is placed at a distance v behind the second lens, shew that the coloured images of this object will appear of the same apparent size to an eye situated at a distance $-v\kappa_2/\kappa_1$ in front of the first lens. (M. T. 1906.)

14. Two thin lenses of the same material, and whose numerical focal lengths are f and f' $(f'>f)$, are placed on an axis at a distance apart equal to the difference of their focal lengths, the former being concave and the latter convex. Shew that if an object is placed at a distance v behind f' its coloured images will all appear of the same angular size to an eye placed at a distance $fv/(f'-v)$ in front of the lens f. (Coll. Exam. 1904.)

15. Shew that if a convex lens be placed between the two convergent lenses (focal length f, distance apart $\frac{2}{3}f$) of a Ramsden's eye-piece so that the distance between them is divided in the ratio $\lambda : 1$, the system will be achromatic for parallel rays if the focal length of the lens be

$$f(\lambda-1)^2/2(\lambda+1)^2,$$

the material of all the lenses being the same. (Coll. Exam. 1905.)

16. Two hemispheres of glass, of index μ and radius r, are placed with their plane surfaces opposed at a distance d. Shew that the combination is achromatic for rays parallel to its axis provided $d = r/\{\mu^2(\mu-1)\}$.

(St John's Coll. 1905.)

17. A sphere of glass of radius r and mean refractive index μ has the central part, lying between two parallel planes, removed, leaving two plano-convex lenses. Prove that for any value of μ not less than $\frac{3}{2}$ the planes may be so chosen that the lens combination is achromatic for rays parallel to its axis, and that the power of the combination is then

$$-4(\mu-1)/r(2\mu+1).$$

(M. T. 1909.)

18. A refracting sphere of radius r and index μ is cut into two hemispheres which are placed with their convex surfaces towards each other. Shew that the combination is achromatic for rays parallel to the axis if the distance between the convex surfaces is

$$r/(\mu-1).$$

(Coll. Exam.)

19. A glass globe of external radius R and thickness t is filled with liquid. Shew that the whole will be achromatic for a direct pencil of parallel rays if the dispersive powers of the glass and liquid are in the ratio

$$\mu^2(\mu'-1)R \text{ to } \mu'^2(\mu-1)t,$$

where μ and μ' are the refractive indices of the glass and the liquid.

(Coll. Exam. 1907.)

20. If two spheres whose radii are r and r', and refractive indices μ and μ', have dispersive powers ϖ and ϖ', and if a be the distance between their centres, shew that the condition of achromatism for a pencil of parallel rays is

$$\frac{\varpi}{\mu}\left(2a-\frac{\mu'r'}{\mu'-1}\right)+\frac{\varpi'}{\mu'}\left(2a-\frac{\mu r}{\mu-1}\right)=0.$$

(Coll. Exam. 1893.)

21. Prove that on the axis of an eye-piece formed of any number of lenses of the same material, there are in general two points (not necessarily real) whose conjugate points are the same for light of all colours.

(Trinity Coll. 1904.)

CHAPTER IX

ILLUMINATION

139. Intensity of Emission.

Light is a form of energy; if we imagine a source at a point to be radiating energy in the form of light waves in all directions uniformly around it, the quantity of light within any cone whose vertex is at the point where the source is situated is clearly proportional to the solid angle of the cone*. If dQ be the quantity of light emitted from any source within a cone of small solid angle $d\omega$, then $dQ/d\omega$ is the **intensity of emission** in the direction of the axis of the cone.

The quantity of light emitted in a given direction from an element of a uniformly bright surface is proportional to the area of the surface and to the intensity of emission in the given direction.

140. It is an experimental fact that a bright surface appears equally bright from whatever angle it is viewed, thus the moon's disc appears to be flat, and if a mass of glowing metal be viewed in a dark room, only the contour and not the shape of the mass will be discernible. From this fact it follows that *the intensity of emission in any given direction from an element of a bright surface is proportional to the cosine of the angle between the given direction and the normal to the surface.*

Imagine a tube EA whose diameter is equal to the pupil of the eye directed obliquely towards a bright surface and let the tube cut the surface in the element AB. An eye at E will receive as much light from AB as from an equally bright surface $A'B$ viewed directly, where $A'B$ is the cross section of the tube at B. And in each case the quantity of light is proportional to the

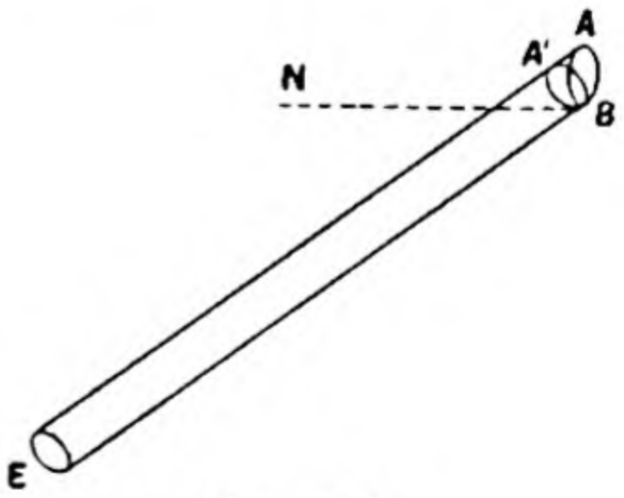

Fig. 85.

* *I.e.* the area intercepted on a sphere of unit radius with its centre at the vertex of the cone.

area of the element and the intensity of emission, hence

$$\frac{\text{intensity of emission in oblique direction } BE}{\text{intensity of emission in normal direction}} = \frac{A'B}{AB} = \cos\theta,$$

where θ is the angle between the direction BE and the normal BN to the bright surface.

141. Intrinsic Brightness.

The intensity of emission in the normal direction at a point of a bright surface is called the **intrinsic brightness** of the surface at the point.

If μ be the intrinsic brightness of a uniformly bright small element of area A, μA is the total emission in the normal direction, and $\mu A \cos\theta$ in a direction making an angle θ with the normal.

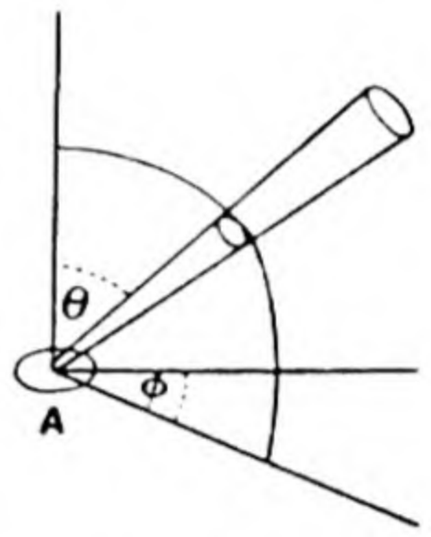

Fig. 86.

Hence the quantity of light within a cone of small solid angle $d\omega$ in this direction is

$$\mu A \cos\theta \, d\omega,$$

or in polar coordinates

$$\mu A \cos\theta \sin\theta \, d\theta \, d\phi.$$

Therefore the total quantity of light emitted in all directions on one side of the element is

$$\mu A \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin\theta \cos\theta \, d\theta \, d\phi = \pi \mu A.$$

142. Illumination.

If dQ be the quantity of light from a given source that falls on an element of area dA then dQ/dA is the **intensity of illumination**.

The strengths of different sources of light can be compared by observing their illuminating effects.

Thus if we consider sources of light to be bright points emitting light uniformly in all directions the quantity of light emitted within a small cone of solid angle $d\omega$ and vertex at the source is $k\,d\omega$, where k is a constant.

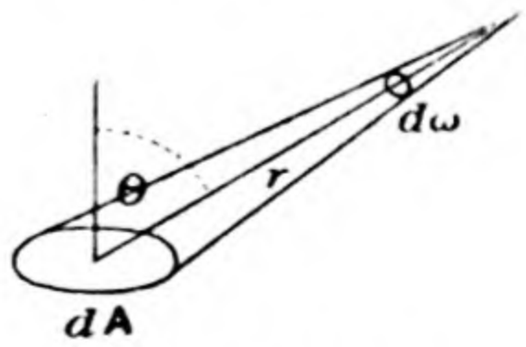

Fig. 87.

Hence the intensity of illumination of an element of area dA which subtends a solid

angle $d\omega$ at the source is $kd\omega/dA$. But $d\omega = dA\cos\theta/r^2$, where θ is the inclination of the direction of the source to the normal to dA and r is the distance.

Hence the intensity of illumination $= \frac{k\cos\theta}{r^2}$, k being a constant which measures the brightness of the source of light.

143. The eye is well able to detect any slight difference in the brightness of two portions of a surface illuminated by monochromatic light, though it cannot compare intensities of brightness. Since intensity of illumination varies directly as the brightness of the source and inversely as the square of the distance, it is easy to effect a comparison of two sources by adjusting them until they produce equal intensities of illumination.

An instrument for this purpose is called a **photometer**. One of the simplest is Bunsen's which consists of a paper screen with a grease spot on it.

If the paper is illuminated from behind, the spot appears brighter than the rest of the paper because it is more translucent. If illuminated from in front it appears darker as it reflects less of the light. If equally illuminated on both sides, the spot and the rest of the paper appear to be of the same brightness.

Hence if two sources of light are placed on opposite sides of the screen and so adjusted that when the light falls directly on the spot from opposite sides it appears of the same brightness as the rest of the paper, then the brightnesses of the sources vary as the squares of the distances from the spot, or

$$k/r^2 = k'/r'^2.$$

144. The unit in terms of which illuminating power is measured should be a standard source of some permanent nature which will not vary when set up at different times or by different observers. Such permanence is difficult if not impossible to attain. The British standard is the **candle-power** which is the illuminating power of a standard sperm candle; but this standard candle is of very variable illuminating power, the variation amounting sometimes to as much as 20 per cent. A more reliable standard is the **Hefner-Alteneck** in which the illuminant is the flame of a lamp in which amyl acetate is burnt.

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145. Illumination produced by a uniformly bright surface.

Let an element of area dA be illuminated by light from an element of area dS of intrinsic brightness μ. If $d\omega$ be the solid angle which dA subtends at a point of dS the quantity of light received by $dA = \mu dS \cos\theta d\omega$ where θ is the inclination of the join of the elements to the normal to dS. But if r is the distance and ϕ the angle that the join makes with the normal to dA

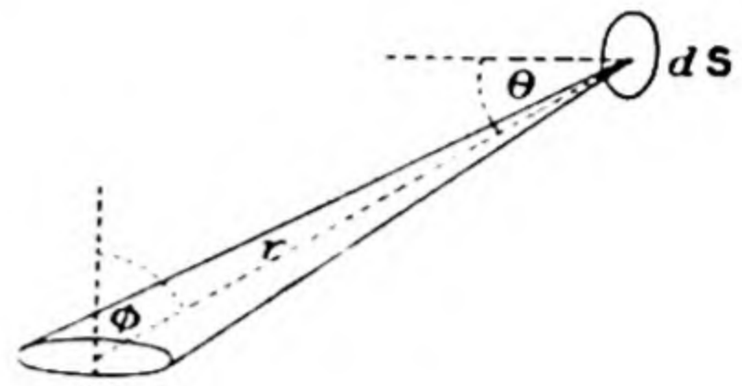

Fig. 88.

$$d\omega = dA \cos\phi / r^2.$$

Therefore the quantity of light received by dA

$$= \mu dS dA \cos\theta \cos\phi / r^2,$$

and the intensity of illumination

$$= \mu dS \cos\theta \cos\phi / r^2.$$

But if $d\varpi$ be the solid angle subtended by dS at a point on dA,

$$d\varpi = dS \cos\theta / r^2.$$

Therefore intensity of illumination $= \mu\, d\varpi \cos\phi$.

Now if a sphere of unit radius be drawn with its centre on dA, the area intercepted on this sphere by the cone of radii drawn to the boundary of dS is $d\varpi$, and $d\varpi \cos\phi$ is the projection of this area on the plane of the element dA. If we denote this projection by $d\sigma$, we have the intensity of illumination of dA produced by dS

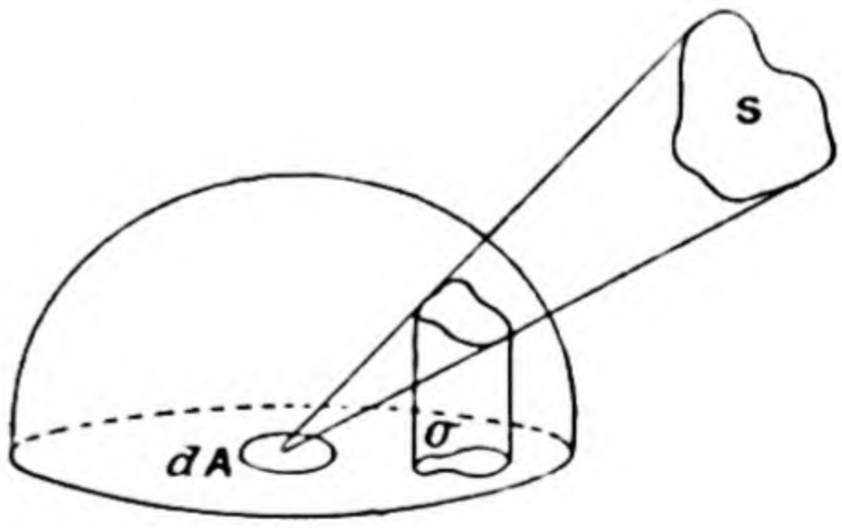

Fig. 89.

$$= \mu d\sigma.$$

The same argument would apply to the illumination produced by every element of a uniformly bright surface S of any form. Hence we get the following rule. *Describe a sphere of unit radius with its centre at the point at which the intensity of illumination is required. Join this point to the boundary of the bright surface S cutting out an area ϖ on the sphere. Project this area ϖ on to the plane of the illuminated element, then if σ be the area of the projection, the intensity of illumination is $\mu\sigma$.*

146. Objects appear equally bright at all distances.

The apparent brightness of an image is measured by the quantity of light from it that falls on the retina divided by the area of the image formed on the retina.

Let A, A' be the areas of the object and its image on the retina, d, d' the distances of the object and image from the optical centre of the eye. The quantity of light that enters the pupil is proportional to $A\varpi/d^2$, where ϖ is the aperture of the pupil, hence the apparent brightness is proportional to $A\varpi/A'd^2$ or ϖ/d'^2, since $A : A' = d^2 : d'^2$. Therefore so long as the size of the pupil remains constant, the apparent brightness is independent of the distance.

The foregoing reasoning ceases to be true when the object is so distant that the image on the retina is indistinguishable from a point. In this case we must take the quantity of light that falls on the retina as the measure of the brightness, and it varies inversely as the square of the distance of the object.

147. Brightness of an image formed by a refracting system.

If l, l' denote the linear dimensions of an object and image placed at right angles to the axis of a coaxial refracting system, α, α' the angles of divergence of a principal ray and μ, μ' the refractive indices of the initial and final media, by Helmholtz's theorem

$$\mu l\alpha = \mu' l' \alpha'.$$

If $d\omega$, $d\omega'$ are the solid angles of corresponding cones of rays diverging from the object and image, we have

$$d\omega : d\omega' = \alpha^2 : \alpha'^2.$$

Also if I, I' denote the intrinsic brightnesses of the object and image, the quantities of light emanating from them within these cones are as

$$Il^2 d\omega : I' l'^2 d\omega',$$

or as

$$Il^2\alpha^2 : I' l'^2 \alpha'^2,$$

i.e. as

$$I/\mu^2 : I'/\mu'^2.$$

But if there be no loss of light by absorption, reflection or refraction, these quantities must be the same, or

$$I/\mu^2 = I'/\mu'^2.$$

And if the initial and final media be the same so that $\mu = \mu'$, then $I = I'$, or the object and image are equally bright. In general there is loss of light, so that the image is not as bright as the object.

148*. Apparent brightness of an object viewed through lenses.

In this case, with the notation of Art. 146, the apparent brightness is proportional to ϖ'/d'^2, where ϖ' is the area of the section of the pencil of light that enters the pupil. Since ϖ' cannot exceed the area of the pupil ϖ, therefore the image cannot appear brighter than the object would appear if viewed directly.

When a star is viewed through a telescope since the image is only a point we must estimate the brightness by the total quantity of light that enters the eye. The quantity that enters the telescope is proportional to the area O of the object-glass, and if the pupil ϖ be greater than the eye-ring (see Art. 174), whose area is O/M^2 where M is the magnifying power, the whole of the light that enters the telescope also enters the eye, and there is an increase of brightness in the ratio $O : \varpi$, due to using the telescope.

But if the pupil be less than the eye-ring, the fraction $\varpi M^2/O$ of the light that enters the telescope also enters the eye, so that the brightness is increased in the ratio $M^2 : 1$ by using the telescope.

To form an estimate of the utility of the telescope, we must also observe its effect on the apparent brightness of the field of view in which the star is situated. This, as we have seen above, depends on the area of the cross section of the pencil that enters the eye. When $\varpi > O/M^2$, the area in question is that of the eye-ring O/M^2 so that the apparent brightness of the field is reduced by the telescope in the ratio $\varpi : O/M^2$; but the brightness of the star was increased in the ratio $O : \varpi$, therefore the increase in brightness of the star as compared with the field is in the ratio $M^2 : 1$.

When $\varpi < O/M^2$, the telescope makes no difference to the apparent brightness of the field, and as before the brightness of the star is in the ratio $M^2 : 1$.

The proper measure of apparent brightness is however really a physiological question, being a matter of subjective impressions.

* The reading of this article may be postponed until after Chap. XII.

EXAMPLES.

1. Two self-luminous spheres of the same intrinsic brightness I and radii a_1 and a_2 are placed at opposite sides of a plane screen so that the line joining their centres is perpendicular to the screen and the distances of their centres from the screen are h_1 and h_2, prove that the locus of points on the screen at which the illumination is the same on both sides of the screen is a circle and find its radius in terms of the given quantities.

If the screen is made of greased paper what is the appearance presented?

(Coll. Exam. 1899)

2. Prove that the illumination at a point on the ground at a distance d from a uniformly bright wall of infinite length and height h, is proportional to

$$1 - d/\sqrt{h^2+d^2}.$$ (Coll. Exam. 1894.)

3. A small area is placed in the horizontal plane at a distance a from the foot of a very long vertical wall of height h. If the wall is uniformly self-luminous, the ratio of the illumination of the area when horizontal to its illumination when vertical is

$$(\sqrt{a^2+h^2} - a)/h.$$ (Coll. Exam. 1891.)

4. A long wall runs along the side of a horizontal area. Shew that the illumination at any point of the area due to a uniformly bright sky varies as $\cos^2 \theta/2$ where θ is the elevation of the wall at its nearest point.

(Coll. Exam. 1907.)

5. Find the illumination at any point of a horizontal plane due to an infinite vertical bright line. (Coll. Exam. 1888.)

6. If the interior of a closed convex surface is uniformly luminous, shew that the additional illumination due to light emitted from each element of the surface and incident on it elsewhere, is the same for every point on the surface.

(Coll. Exam. 1906.)

7. Shew that the illumination at a point on the ground due to the upper half of the Sun's disc is to that due to the lower half as

$$\pi \sin a \sin^2 \beta + 2 \cos a \,(\beta - \sin \beta \cos \beta)$$

is to

$$\pi \sin a \sin^2 \beta - 2 \cos a \,(\beta - \sin \beta \cos \beta),$$

where a is the altitude of the Sun's centre, 2β the Sun's angular diameter.

(Coll. Exam.)

8. If a horizontal plane and a very long and very high vertical straight wall are illuminated by the sky and one another, prove that on the assumption that the brightness of a point on either wall or plane is k (small) times its illumination, the brightness of a point on the plane near the wall is approximately

$\frac{1}{2}\pi k\,(1 + \frac{1}{2}\pi k)$ times

that of the sky. (Coll. Exam. 1900.)

9. Two uniformly bright vertical walls of the same uniform height and infinite length intersect at right angles, shew that the illumination at a point on the ground is

$$\frac{3\pi\mu}{4} - \frac{\mu}{2}\cos\alpha\{\pi - \tan^{-1}(\operatorname{cosec}\alpha\tan\beta)\} - \frac{\mu}{2}\cos\beta\{\pi - \tan^{-1}(\operatorname{cosec}\beta\tan\alpha)\},$$

where μ is the intrinsic brightness of the walls, and α and β are the greatest angles subtended at the point by vertical lines drawn on the walls.

(Magd. Coll. 1898.)

10. A uniformly bright elliptic disc is placed with an extremity of its major axis on a horizontal table, the axis itself being vertical. Find at what points on the table it will appear to be a circular disc, and prove that at these points the intensity of illumination of the table is

$$\pi\mu e^3/(1+e^2)^{\frac{3}{2}},$$

where e is the eccentricity of the ellipse and μ is the intrinsic brightness.

(M. T. 1900.)

11. An infinitely long self-luminous circular cylinder of uniform brightness touches a plane; shew that the illumination at any point of the plane varies inversely as the square of the distance of the point from the axis of the cylinder.

(M. T. 1898.)

12. An infinite plane surface is divided by a parabola of focus S and vertex A, and the portion containing the focus is uniformly bright. If P is a point on the parabola whose vertex is S and focus A and whose plane is perpendicular to that of the first parabola, prove that the intensity of illumination of a small plane area at P normal to the locus of P varies inversely as AP.

(Coll. Exam. 1904.)

13. If a bright surface be a square and the line joining the centre of the square to a point on a plane surface be normal to both surfaces, shew that the illumination at the point is proportional to $\sin\alpha\tan^{-1}(\sin\alpha)$, α being the angle subtended at the point by the line joining the centre of the square to the middle point of a side.

(Coll. Exam. 1893.)

14. A small area is placed in any way at the centre of a regular polyhedron: shew that it is equally bright on both sides, the vertices of the solid being luminous.

(St John's Coll. 1897.)

15. A horizontal area at a point O is illuminated by a bright sky, whose intrinsic brightness at any point P is equal to $\mu\cos^2\frac{1}{2}AOP$, A being a fixed point in the sky. Shew that the illumination at O is equal to

$$\mu\pi\left(\tfrac{1}{2} + \tfrac{1}{3}\cos z\right)$$

where z is the angular distance of the point A from the zenith of O.

(Coll. Exam. 1902.)

16. A luminous sphere is inside a spherical shell. Shew that the points of maximum and minimum illumination are at the extremities of the line joining their two centres; and if the ratio of the maximum to the minimum illumination be given, find the distance between the centres of the spheres.
(Coll. Exam.)

17. Five of the internal faces of a cubical box are uniformly bright; shew that the intensity of illumination at all points of the sixth face is the same, and that at the centre of this face the intensity of illumination due to the opposite face is to the whole intensity as

$$\frac{4}{\sqrt{5}} \cos^{-1} \sqrt{\frac{5}{6}} : \pi.$$ (Coll. Exam. 1901.)

18. A candle flame is placed at a point on the axis of a divergent lens of focal length 6 in., at a distance of 12 in. from the lens; shew that the illumination of a screen at a point on the axis at a distance of 8 in. from the lens is less than it would be if the lens were absent in the ratio 25 : 81.
(M. T. 1906.)

19. Explain why stars appear brighter when viewed through a telescope. Examine whether objects subtending a finite angle to the observer also appear brighter when viewed through a telescope. (Coll. Exam. 1907.)

20. Prove, on the assumption that there is no loss of light in passing the lenses, that the brightness of an object which has a finite angular magnitude cannot be increased by the use of a telescope and that it may be diminished.

Explain the advantage of high magnifying power when a telescope is used for viewing faint stars. (Coll. Exam. 1904.)

CHAPTER X

MISCELLANEOUS THEOREMS

149. *When a ray of light traverses several media undergoing any number of refractions and reflections, the path from one point to another is such that* $\Sigma\mu\rho$ *is stationary in value for small variations in the path, where* ρ *is the length of the path in the medium of refractive index* μ*.*

It is clearly sufficient to prove this theorem for a single refraction.

Let CD be an interface between two media of refractive indices μ, μ' and APB the path of a ray from A to B crossing the interface at P.

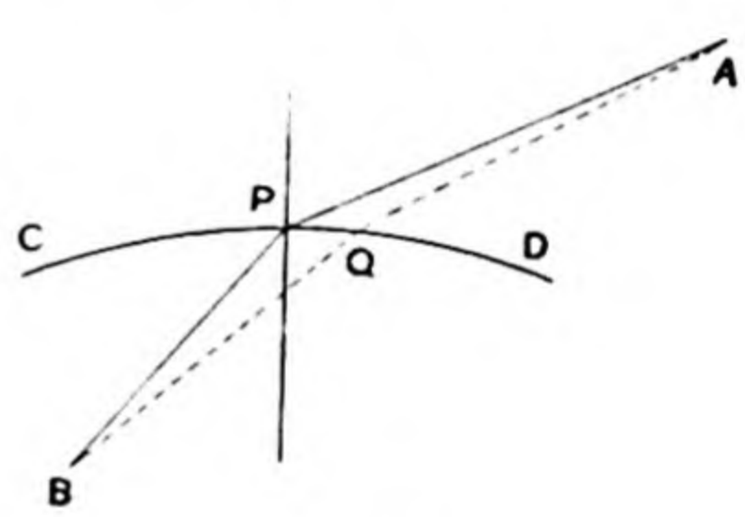

Fig. 90.

We have to shew that $\mu AP + \mu' PB$ is stationary for the actual path. Let Q be any point on CD near to P, and let α, α' be the complements of the angles APQ, BPQ, then

$$\mu AQ + \mu' QB - (\mu AP + \mu' PB)$$
$$= \mu (AQ - AP) + \mu' (QB - PB)$$
$$= -\mu PQ \sin\alpha + \mu' PQ \sin\alpha' = 0. \quad \text{(Art. 48.)}$$

Hence the expression $\mu AP + \mu' PB$ is stationary in value for small displacements of the point P.

The case of a reflection follows by putting $\mu' = -\mu$ or may be proved independently in the same way.

For any number of refractions or reflections, since the expression $\Sigma\mu\rho$ is stationary for small displacements of each point of incidence taken separately, therefore it is also stationary for any small variations of the path from one point to another.

* First enunciated by P. Fermat : *Litterae ad P. Mersenum contra Dioptricam Cartesianam*, Paris, 1667.

If the path be through a medium of continuously varying refractive index, the theorem takes the form: $\int \mu ds$ is stationary for small variations in the path from one point to another.

The expression $\Sigma \mu \rho$ or $\int \mu ds$ is frequently called the *reduced path*.

Remembering that the refractive index of a medium varies inversely as the velocity of light in that medium, we see that the foregoing theorem merely expresses that the time taken for light to travel from one point to another is stationary in value for small variations of the path.

150. Theorem of Malus.

Any bundle of rays which can be cut orthogonally by a surface will retain this property after any number of reflections and refractions.*

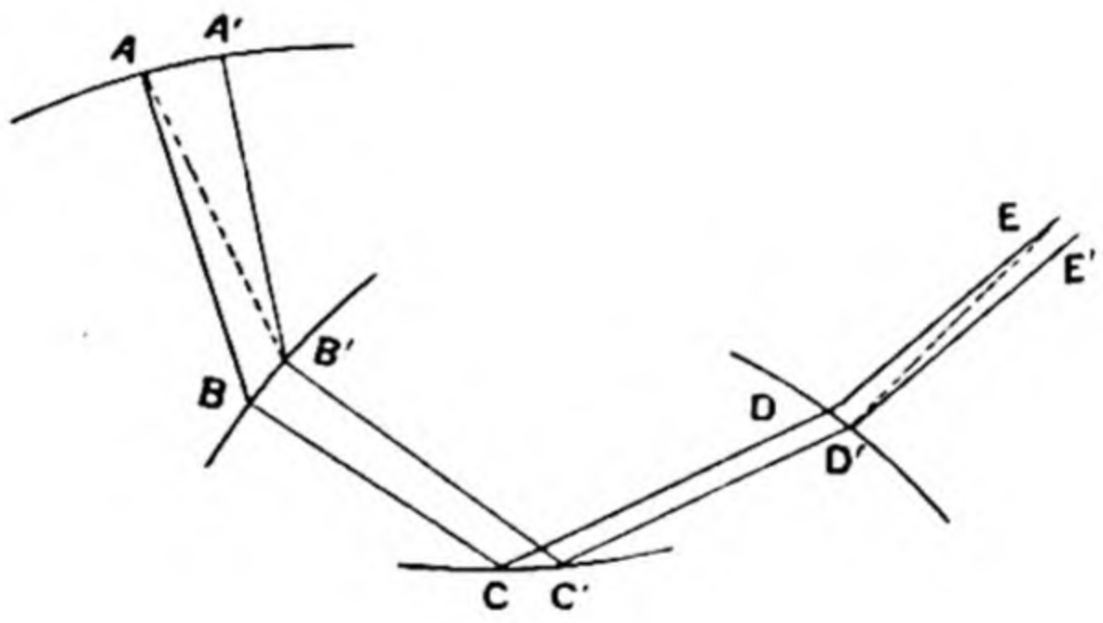

Fig. 91.

Let $ABCDE, A'B'C'D'E' \ldots$ be a bundle of rays normal to a surface $AA' \ldots$.

Along the rays measure off lengths to points $E, E' \ldots$ such that the reduced path $\Sigma \mu \rho$ is the same for each ray.

Join AB', ED'. Then

$$\text{reduced path } (A'B'C'D'E') = \text{reduced path } (ABCDE)$$
$$= \quad \text{,,} \quad \text{,,} \quad (AB'C'D'E),$$

since the reduced path from A to E is stationary in value.

Removing the common part, this leaves

$$\mu A'B' + \mu' D'E' = \mu AB' + \mu' D'E$$

if μ, μ' are the refractive indices of the initial and final media. But since the rays are cut at right angles by AA', we have

* E. L. Malus: "Optique," *Journal de l'École Polytech.* VII. p. 1, 1808.

$A'B' = AB'$ ultimately, and therefore $D'E' = D'E$, or EE' is ultimately at right angles to $D'E'$.

The same property holds good for all points adjacent to E, E' determined in the same way. Therefore there is a surface through E, E' orthogonal to the ray $D'E'$, and also by a similar argument to DE and to every ray of the system.

151. A system of rays which can be cut orthogonally by a surface is called an **orthotomic system**. A system radiating from a point is clearly orthotomic, for it is cut orthogonally by any sphere which has the point for centre.

In the wave theory of light the orthotomic surfaces are the wave-fronts.

If a system of rays diverge from a point and after any number of reflections or refractions converge to another point, the value of $\Sigma\mu\rho$ must clearly be the same for all the rays.

Thus to determine the form of a surface which will reflect rays diverging from a point S so as to pass after reflection through a point H, we must have $SP + PH = \text{const.}$, where P is any point on the surface. The surface is therefore a prolate spheroid, with S, H as foci of a meridian section.

To reflect as parallel rays a pencil diverging from a point, we must have a mirror in the form of a paraboloid of revolution, for the second focus H is now at an infinite distance.

To refract rays diverging from S so as to make them pass through H we must have

$$\mu SP + \mu' PH = \text{const.},$$

which represents surfaces formed by revolving Cartesian ovals about the line joining the foci S and H.

152. Focal length in terms of aperture and thickness.

As an application we may obtain the focal length of a convergent lens in terms of its aperture and thickness.

Let a beam of parallel rays fall on the lens and converge to a focus F. Since the plane RPQ is at right angles to the incident rays, the reduced paths PDF and RAF must be equal. Hence, if $AC = y$

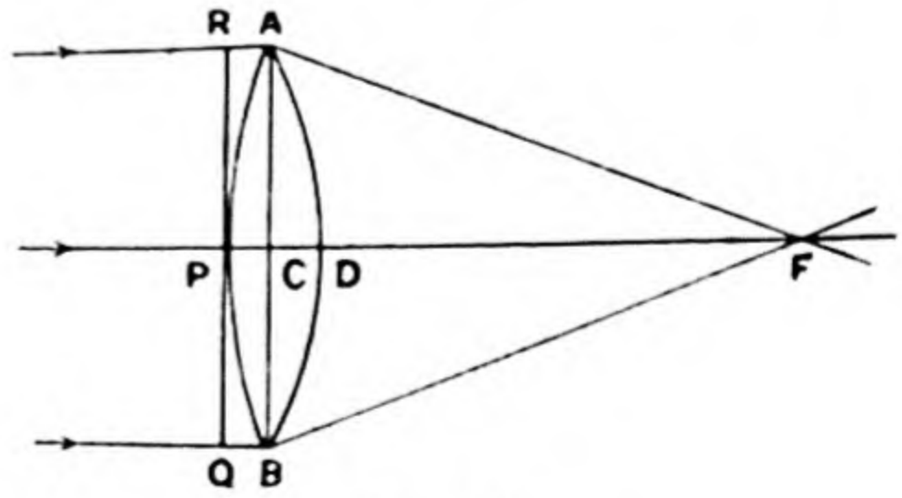

Fig. 92.

be the semi-aperture, $PD = t$ be the thickness, and $CF = f$ be the focal length,

$$\mu PD + DF = RA + AF,$$

or
$$\mu t + f - CD = PC + \sqrt{f^2 + y^2},$$

therefore $(\mu - 1)\, t = \frac{1}{2}\, y^2/f$ to the second order of y/f.

Hence
$$f = \frac{1}{2}\, \frac{y^2}{(\mu - 1)\, t}\ *.$$

153. Caustics.

When an instrument is symmetrical about an axis and a pencil of rays diverges from a point on the axis, the symmetry about the axis will persist after any number of refractions; and the orthotomic surfaces will be surfaces of revolution about the axis.

The rays in any meridian plane are normals to a curve, the meridian section of an orthotomic surface, and as the intersections of consecutive normals to a curve form another curve or evolute of the given curve, which is touched by the normals, it follows that the rays in any meridian plane envelope a curve, called a **caustic curve**, which is the evolute of an orthotomic curve. Also, all the rays touch a **caustic surface** generated by the revolution about the axis of the caustic curve.

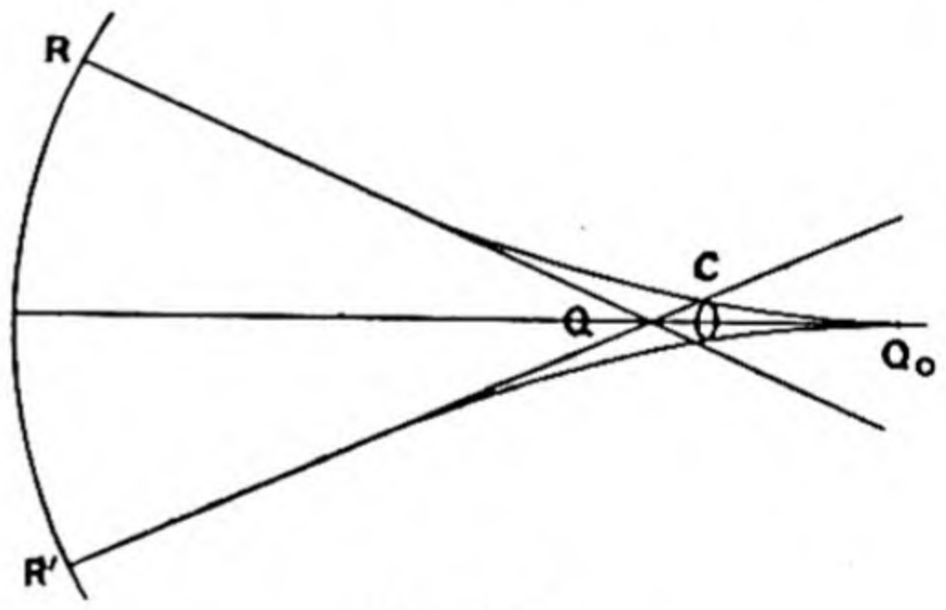

Fig. 93.

154. If Q_0 be the cusp on the caustic curve, *i.e.* the geometrical focus of the pencil of rays, and if a ray of the pencil cut the axis in Q, then QQ_0 is called the **longitudinal aberration** of the ray. If α be the inclination of the ray RQ to the axis, we may assume that the aberration QQ_0 is proportional to α^2; calling it $c\alpha^2$ and taking Q_0 as origin and Q_0Q as axis of x, the equation of the ray QR is

$$y = (x - c\alpha^2) \tan \alpha,$$

or, to the third order of α,

$$y = x\alpha - c\alpha^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The caustic curve is the envelope of this line, α being a variable parameter. Differentiating with regard to α, we get

$$0 = x - 3c\alpha^2.$$

* Lord Rayleigh, *Phil. Mag.* (5), VIII. p. 480, 1879, or *Scientific Papers*, Vol. I. p. 439.

Therefore the point of contact of the ray with the envelope is

$$(3c\alpha^2,\ \ 2c\alpha^3),$$

and the caustic curve, in the neighbourhood of its cusp, is the semi-cubical parabola

$$27cy^2 = 4x^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

155. Least circle of aberration.

If a screen be moved along at right angles to the axis, the rays will illuminate a circular patch on the screen and this circle will clearly be least when the screen is at C where the extreme rays of the pencil intersect the caustic. This circle is called the **least circle of aberration.**

To find its position, let α be the inclination of the extreme ray, then eliminating y from (1) and (2) we get

$$27c\,(x\alpha - c\alpha^3)^2 = 4x^3,$$

a cubic equation in x, two of whose roots are $3c\alpha^2$; but the product of the roots is $27c^3\alpha^6/4$, therefore the third root is $\frac{3}{4}c\alpha^2$. Hence the distance of the least circle of aberration from the geometrical focus is $\frac{3}{4}$ the longitudinal aberration of the extreme ray, and the radius of the circle is $\frac{1}{4}c\alpha^3$.

156. The **lateral aberration** of a pencil is the radius of the pencil as it passes the geometrical focus Q_0. To find it, we put $x = 0$ in (1), and find that the lateral aberration is $c\alpha^3$, so that the radius of the least circle of aberration is $\frac{1}{4}$ of the lateral aberration.

157. When rays from a point are refracted at a straight line, the caustic is the evolute of an ellipse.

Let SPR be a ray refracted at a point P on the straight line XY. Let SC at right angles to XY be produced to H making $CH = SC$. Let the refracted ray meet the circle SPH in Q.

If ϕ, ϕ' are the angles of incidence and refraction, we have

$$PSH = PQH = PQS = PHS = \phi.$$

Hence if d be the diameter of the circle,

$$SQ = d\sin(\phi' - \phi),$$

and

$$HQ = d\sin(\phi' + \phi),$$

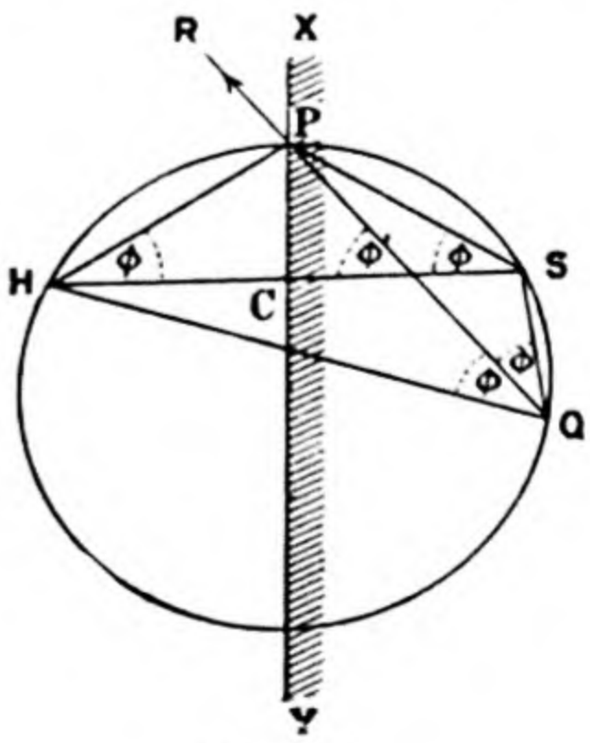

Fig. 94.

therefore

$$SQ + HQ = 2d \sin \phi' \cos \phi = \frac{d}{\mu} \sin 2\phi$$
$$= \frac{1}{\mu} SH.$$

Therefore Q lies on an ellipse with S and H as foci and of eccentricity μ. Also the ray QPR bisects the angle between the focal distances and is therefore normal to the ellipse at Q. The caustic is therefore the evolute of this ellipse.

158. Images.

It is important to distinguish between the geometrical image of an object, or the locus of geometrical foci, and the visible image which alters with the position of the eye and is found by drawing tangents from the eye to the caustic curves which correspond to the various points of the object.

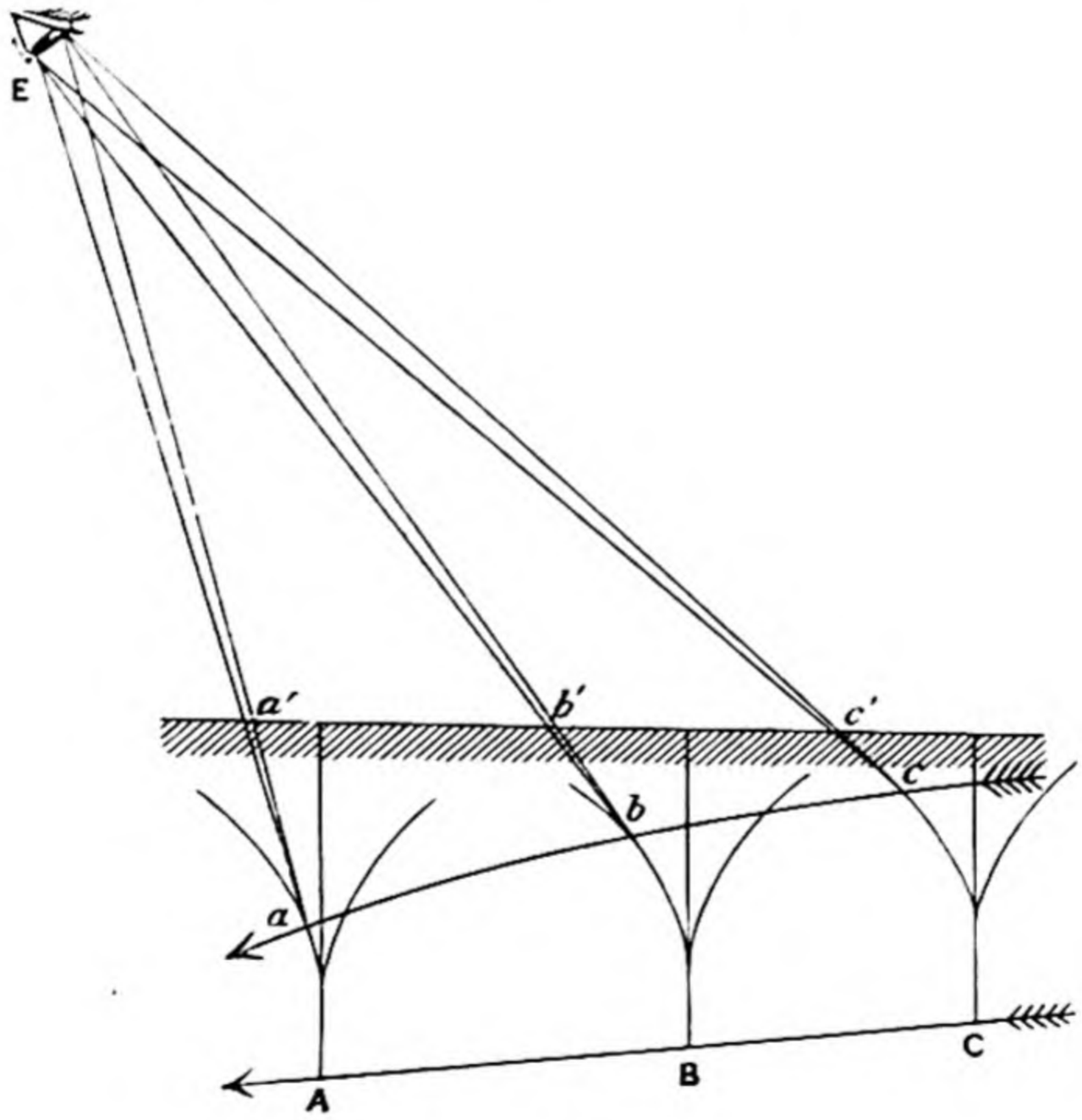

Fig. 95.

The figure shews the way in which a rod ABC under water is viewed by an eye at E. The caustics of the various points A, B, C are evolutes of ellipses. The two tangents from the eye to each caustic shewn in the figure intersect very nearly in points a, b, c lying on the caustics and these points a, b, c are the points of the visible image corresponding to A, B, C.

The actual paths of the rays are $Aa'E$, $Bb'E$, $Cc'E$, where a', b', c' are points in which the tangents to the caustics meet the surface of the water.

159. Thin pencils. Focal lines.

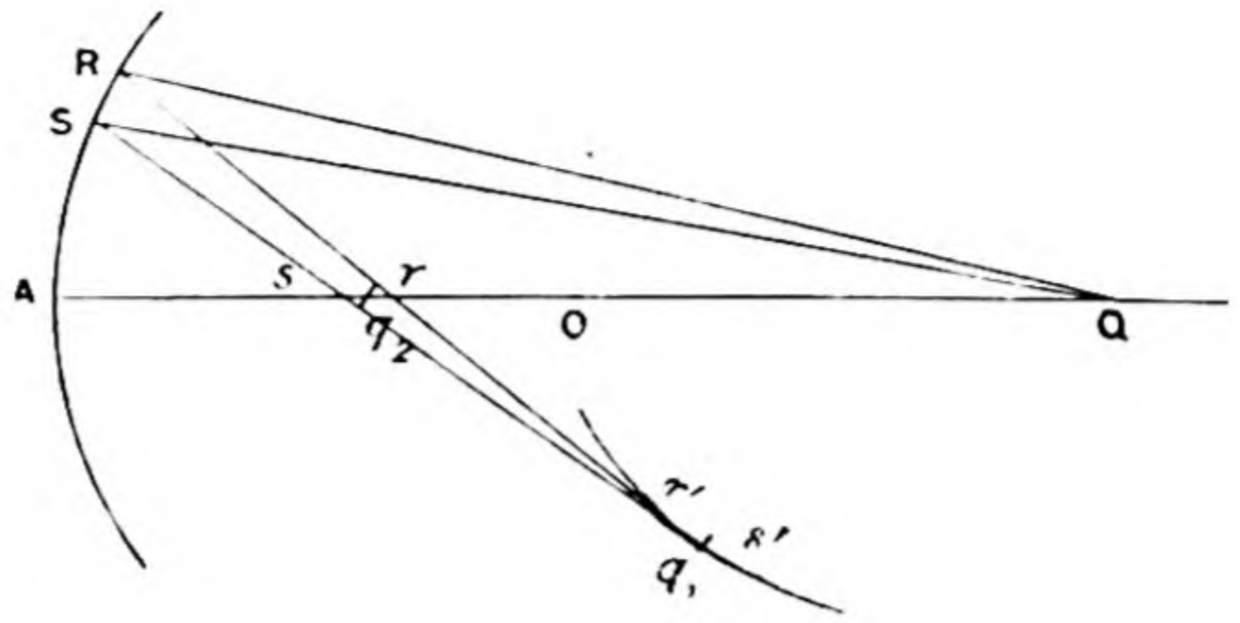

Fig. 96.

Limiting our consideration to the case of symmetry about an axis, suppose a thin pencil from a point Q to be reflected or refracted at a spherical surface, of which the figure represents a meridian section. Let QR, QS be the extreme rays in this section and after reflection or refraction let the rays cut the axis QOA in r, s and touch the caustic curve in r', s'. All rays between R, S will cut the axis between r, s and touch the caustic curve between r', s'. If the arc RS be small compared to RQ or Rr', we may regard the reflected or refracted rays in the meridian section as all passing through the same point q_1 in which Rr', Ss' intersect. Now rotate the figure through a small angle about the axis. The arc RS traces out a small patch on the sphere, the point q_1 traces a small arc of a circle, approximately a straight line at right angles to the meridian plane, and all rays of the pencil incident on the patch referred to will pass through this short straight line through q_1. This straight line is called the **primary focal line**, and q_1 is the **primary focus**, and the meridian plane the **primary plane**.

Again, the central ray of the pencil passes through a point q_2 on the axis, between r and s. Since every ray of the pencil intersects the line rs, it follows that the cross section of the pencil at q_2 is an elongated figure, approximately a straight line in the primary plane. This line is called the **secondary focal line** and q_2 is the **secondary focus***.

* The theory of focal lines is due to J. C. Sturm: "Mémoire sur l'optique," *Liouville's Journal*, t. III. p. 357, 1838.

160. To find formulae for the Primary and Secondary Foci.

1. *Reflection at a spherical mirror.*

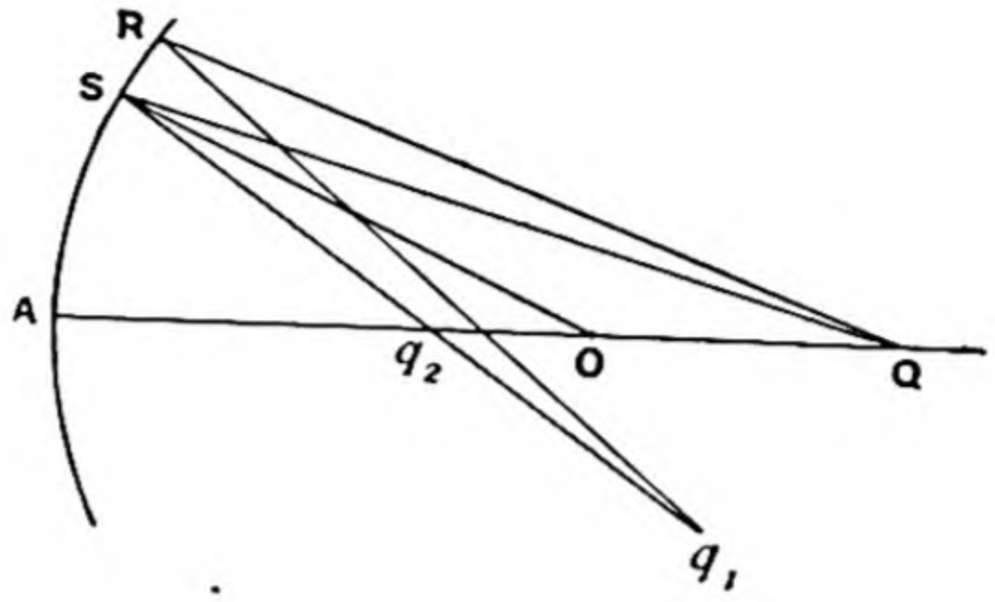

Fig. 97.

Let $SQ = u$, $Sq_1 = v_1$, $Sq_2 = v_2$, $SOA = \theta$, $q_2SO = OSQ = \phi$; and let r be the radius.

Then $$\theta - \phi = OQS \text{ and } SR = r\,d\theta,$$

therefore

$$d(\theta - \phi) = SQR = SR\cos\phi/u = r\,d\theta\cos\phi/u \ldots\ldots\ldots(1).$$

Also $$\theta + \phi = Aq_2S,$$

therefore

$$d(\theta + \phi) = Sq_1R = SR\cos\phi/v_1 = r\,d\theta\cos\phi/v_1 \ldots\ldots\ldots(2).$$

Hence by adding (1) and (2)

$$\frac{1}{v_1} + \frac{1}{u} = \frac{2}{r\cos\phi} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Again $$\frac{r}{v_2} = \frac{\sin(\theta + \phi)}{\sin\theta} = \cos\phi + \cot\theta\sin\phi,$$

and $$\frac{r}{u} = \frac{\sin(\theta - \phi)}{\sin\theta} = \cos\phi - \cot\theta\sin\phi.$$

Hence by addition

$$\frac{1}{v_2} + \frac{1}{u} = \frac{2\cos\phi}{r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

2. *Refraction at a plane surface.*

This case has been dealt with in Art. 34.

3. *Refraction at a spherical surface.*

With the same notation, let $OSQ = \phi$, $OSq_1 = \phi'$ (Fig. 98).

Then $$\theta - \phi = OQS \text{ and } SR = r\,d\theta;$$

therefore

$$d(\theta - \phi) = SQR = SR\cos\phi/u = r\,d\theta\cos\phi/u \ldots\ldots\ldots(1).$$

Also $$\theta - \phi' = Oq_2S;$$

therefore $$d(\theta - \phi') = Sq_1R = SR\cos\phi'/v_1 = r\,d\theta\cos\phi'/v_1 \ldots\ldots(2).$$

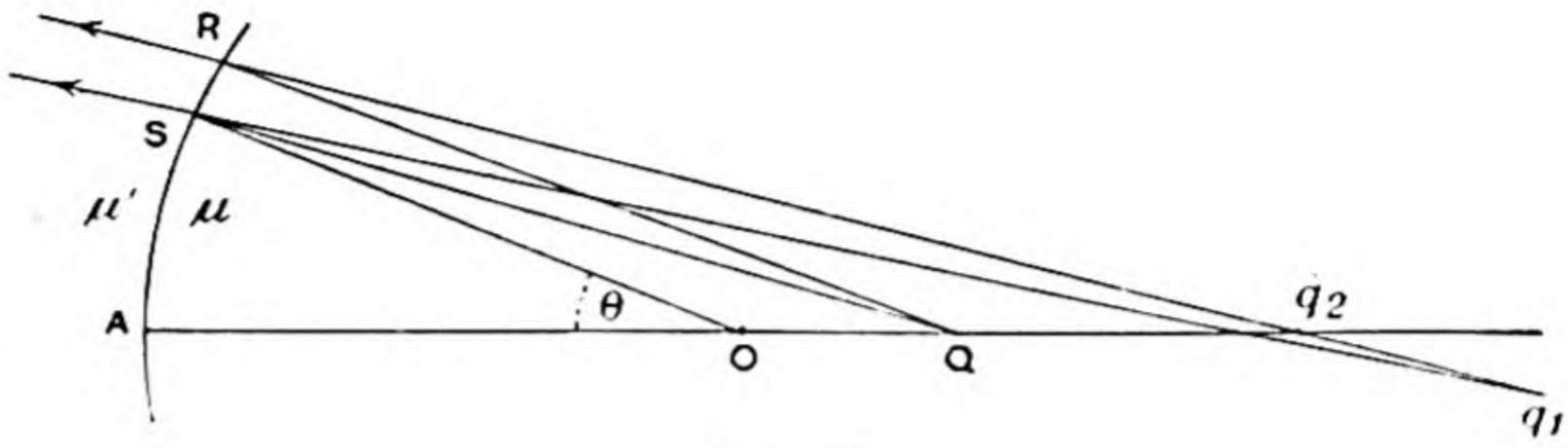

Fig. 98.

But $$\mu\sin\phi = \mu'\sin\phi'$$

so that $$\mu\cos\phi\,d\phi = \mu'\cos\phi'\,d\phi' \ldots\ldots(3).$$

Hence from (1), (2) and (3)

$$\frac{\mu\cos^2\phi}{u} - \frac{\mu'\cos^2\phi'}{v_1} = \frac{\mu\cos\phi - \mu'\cos\phi'}{r} \ldots\ldots(4).$$

Again $$\frac{r}{v_2} = \frac{\sin(\theta - \phi')}{\sin\theta} = \cos\phi' - \cot\theta\sin\phi',$$

and $$\frac{r}{u} = \frac{\sin(\theta - \phi)}{\sin\theta} = \cos\phi - \cot\theta\sin\phi.$$

Therefore $$\frac{\mu}{u} - \frac{\mu'}{v_2} = \frac{\mu\cos\phi - \mu'\cos\phi'}{r} \ldots\ldots(5).$$

161. Circle of least confusion.

Since the cross sections of the thin pencil at the primary and secondary foci are approximately straight lines at right angles to one another, the planes through these lines being the secondary and primary planes respectively, it follows that there will be a point between the focal lines where the breadth of the pencil in the primary and secondary planes will be the same. If we assume that the cross section in this position is a circle, it is called the **circle of least confusion.** Its position depends on the form of the pencil.

MISCELLANEOUS EXAMPLES.

1. Light is incident nearly normally from a source A on a thick plane mirror at P, is refracted at P, reflected at the back at Q, and reaches the front surface again at R; prove that the reduced path $APQR$ differs from AR by

$$2\mu t - t\rho^2/a\,(\mu a + 2t),$$

where t is the thickness of the mirror, a is the perpendicular AM from A on the front, and ρ is the distance MR. (St John's Coll. 1914.)

2. A and B are the extreme lenses of an optical instrument symmetrical about an axis and of length l, which is such that the image of every object formed by it is erect and at a constant distance $l+d$ from the object. Lenses of focal lengths f_1 and f_2 are placed in contact with the lenses A and B respectively. Prove that if $f_1+f_2=d$, the power of the instrument is zero and the linear magnification of the image of an object placed on the axis beyond the end B is $-f_1/f_2$. (M. T. 1906.)

3. If a series of smooth grooves be cut on a plane surface in the form of concentric circles, and light be reflected at them coming from a luminous point, prove that a bright curve in the form of a circle will be observed by an eye situated in the same plane as the luminous point and the axis of the circles. (Coll. Exam. 1895.)

4. When a small object is placed at A on the axis of an instrument consisting of any number of coaxial spherical refracting surfaces, it is found that a real image is formed at B; at A and B two plane mirrors are placed at right angles to the axis; prove that if a small object be placed near A on the axis between the mirrors, there will be formed between A and the instrument a series of images whose distances from A form a harmonical progression, and whose linear dimensions perpendicular to the axis are in the same ratio as their distances from A. (M. T. 1896.)

5. A ray of light is refracted through a prism in a principal plane. Shew that, if the dispersion of two neighbouring colours be a minimum,

$$\frac{\sin(3\phi'-2i)}{\sin\phi'}=1-\frac{2}{\mu^2}.$$ (Coll. Exam. 1897.)

6. Find the equation to the caustic by reflection, at a circle, of rays proceeding from a luminous point in the plane of the circle. Shew that if the luminous point is anywhere outside the circle, the caustic touches the circle at two points. (Coll. Exam. 1906.)

7. Rays issuing from the cusp are reflected at the cardioid whose equation is $r=a(1-\cos\theta)$. Shew that the caustic is the evolute of the curve whose equation may be put in the form $r=4a\sin^3\frac{\theta}{3}$. (St John's Coll. 1884.)

8. A pencil of rays is incident on a three-cusped hypocycloid in a direction parallel to the tangent at one of the cusps; prove that the caustic after reflection is a four-cusped hypocycloid. (M. T. 1904.)

9. An eye, situated above the surface of still water, observes a small linear object lying in a horizontal plane, in a direction at right angles to the plane of incidence and refraction of the thin pencil by means of which the object is seen, the axis of the pencil making an angle ϕ with the vertical. Shew that the apparent depth h' is connected with its real depth h by the equation

$$(\mu^2 h/h')^{\frac{2}{3}}=1+(\mu^2-1)\sec^2\phi,$$

μ being the coefficient of refraction from air into water. (Coll. Exam. 1905.)

10. A plane area is placed perpendicular to the axis of a concave spherical reflector; shew that the surface, on which the geometrical foci lie, is a portion of a spheroid of revolution, and that the curvature at the vertex of the generating conic is independent of the distance of the plane from the centre of the sphere. (Coll. Exam. 1899.)

11. Explain how a convex lens operates by delaying the central part of the wave-fronts of the light compared with the peripheral parts: and hence or otherwise prove the relation between the distances of conjugate foci for a thin lens

$$\frac{1}{v} + \frac{1}{u} = (\mu - 1)\frac{2t}{y^2},$$

where $2y$ is the breadth of the lens and t is the excess of its thickness at the centre over its thickness at the edge. Express the focal length of the lens in terms of μ, t and y.

Prove that the more general formula for the central zone of a lens, made of material whose refractive index gradually changes from the centre outwards according to the law

$$\mu = \mu_0 + \frac{y^2}{c^2}, \quad \text{is} \quad \frac{1}{v} + \frac{1}{u} = -\frac{d^2}{dy^2}\{(\mu - 1)\, t\},$$

where t is the thickness at distance y from the axis; and hence shew that the power of the lens is decreased, beyond the value due to the curvatures of the faces, by $2t_0/c^2$, where t_0 is its thickness at the centre. (M. T. ii. 1911.)

CHAPTER XI

THE EYE AND VISION

162. The eye.

The human eye is an optical instrument formed of refracting media bounded by curved surfaces.

It is nearly spherical in form, the front part projecting slightly.

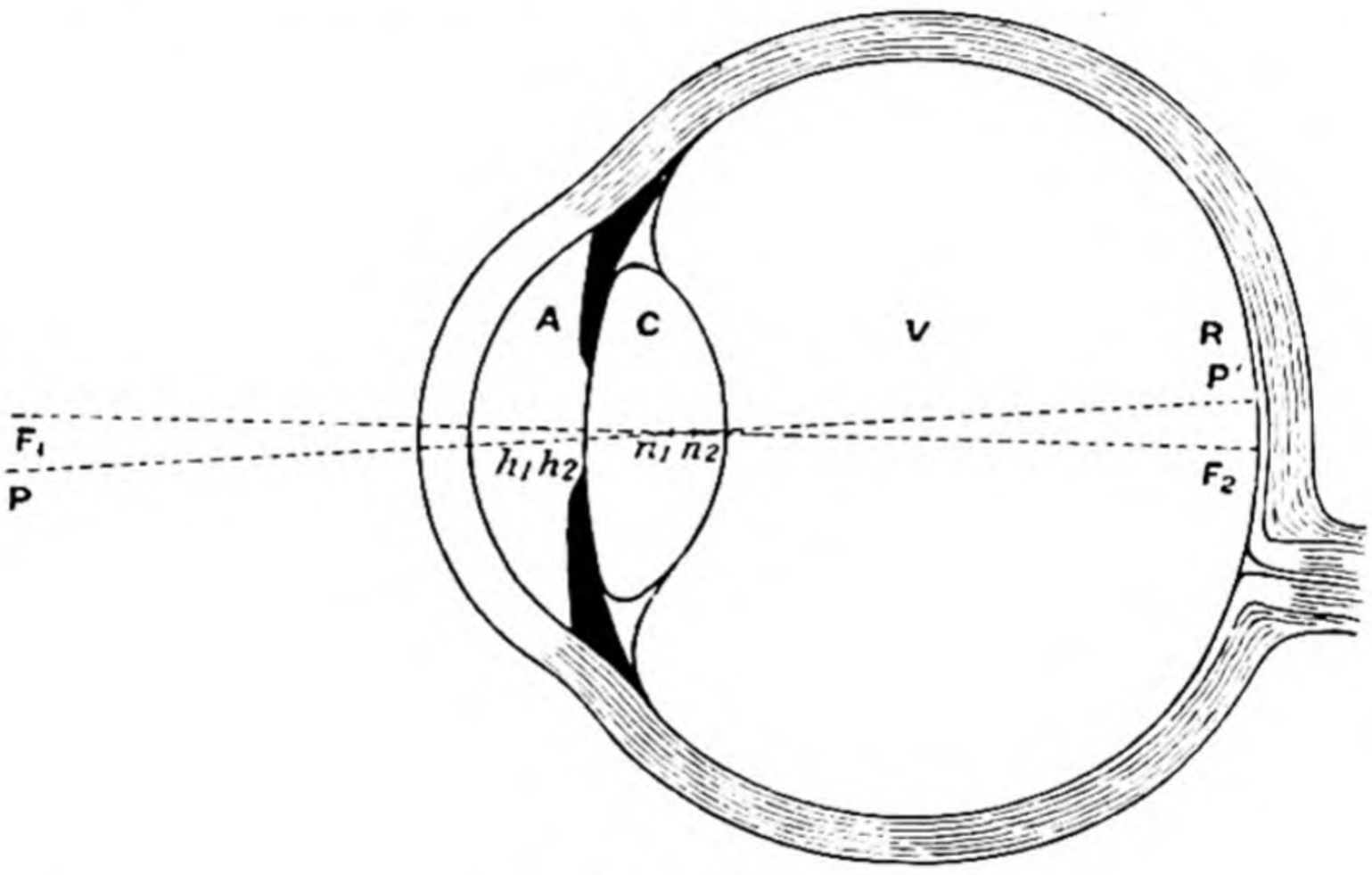

Fig. 99.

It is enclosed in a thick coating called the *sclerotic* which is opaque except in front where it is transparent, this protuberant transparent coating is called the *cornea*. Immediately inside the sclerotic is a thin membrane called the *choroid*, in front the choroid is thickened and forms a diaphragm called the *iris* with an aperture called the *pupil* capable of contraction.

Behind the iris is the *crystalline lens*; this is a convergent lens, with both surfaces convex, the front surface being less curved than the back one. Between the cornea and the crystalline lens is a colourless liquid called the *aqueous humour*. The eye has an inner lining called the *retina* consisting of a layer of nerves formed by the spreading out of the optic nerve which communicates with the brain. Between the crystalline lens and the retina is a gelatinous mass known as the *vitreous humour*.

In the centre of the retina is a round yellowish spot known as the *yellow spot*, and vision is most distinct when the image of an object is formed on the yellow spot. A small distance at one side of the yellow spot is the *blind spot* from which the fibres of the optic nerve spread out to form the retina. This portion is not sensitive to light and an image formed on it is not visible.

The figure shews a horizontal section of the eye, in which A, C, V, R are the aqueous humour, crystalline lens, vitreous humour and retina respectively.

In normal vision when rays from a point of a distant object enter the eye, they are brought to a focus on the retina and an inverted image is formed there, but the sensation produced in the brain is the appearance of an erect object.

The surfaces of the crystalline lens have radii of curvature approximately 10 mm. and 6 mm. and these vary slightly in order to enable the eye to form images on the retina of objects at different distances. This power of varying the curvatures is known as *accommodation*. There is a limit to the amount of accommodation that is possible, and a distance from the eye within which objects are not clearly visible. This distance is called the *least distance of distinct vision*.

The cardinal points of the eye are shewn in the figure, the distance between the unit points h_1, h_2 or between the nodal points n_1, n_2 varies between ·3 and ·4 mm. according to the adjustment of the eye, so that they are almost coincident.

The angle that an object subtends at n_1 is equal to that which its image on the retina subtends at n_2. This is the measure of angular magnitude formed by the eye. Consequently when we speak of *the eye* being at a certain point, it is the first nodal point of the eye whose position is referred to.

163. Formation of images on the retina.

Light entering the eye is refracted first at the cornea, then when it enters the aqueous humour, next at the front of the crystalline lens and finally as it passes from the crystalline lens to the vitreous humour. The refractive indices of the cornea and the humours are so

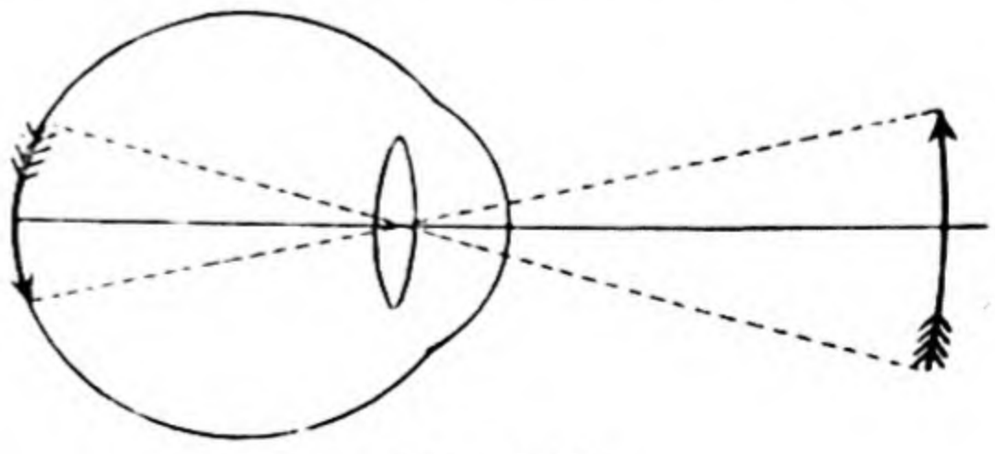

Fig. 100.

nearly equal that the refraction from the cornea to the aqueous humour may be neglected and the lens may be regarded as having the same medium on both sides of it. The combination is a convergent one, and the images of objects beyond the first focus of the eye are inverted images on the retina and are reinverted by the brain. This fact may be shewn experimentally by admitting to the eye pencils diverging from a point very near to the eye; thus, if three pinholes are made at the corners of a small triangle ABC on a piece of cardboard and the cardboard be held with the pinholes close to the eye with A uppermost and BC horizontal, and light be admitted through the pinholes by holding near to them another card with a single pinhole in it, it is clear that there will be three small patches of light on the retina with A uppermost, but the impression formed by the brain is that the vertex A is below BC.

164. Defects of Vision.

(1) **Myopia or Short Sight.** This means that the eye cannot see distant objects clearly, because the eyeball is too long and images are formed between the crystalline lens and the retina instead of on the retina. Another way of stating the cause of this defect is to say that the eye is too strongly convergent, or as Newton said "too plump," and consequently brings parallel rays to a focus in front of the retina. Nearer objects are seen more

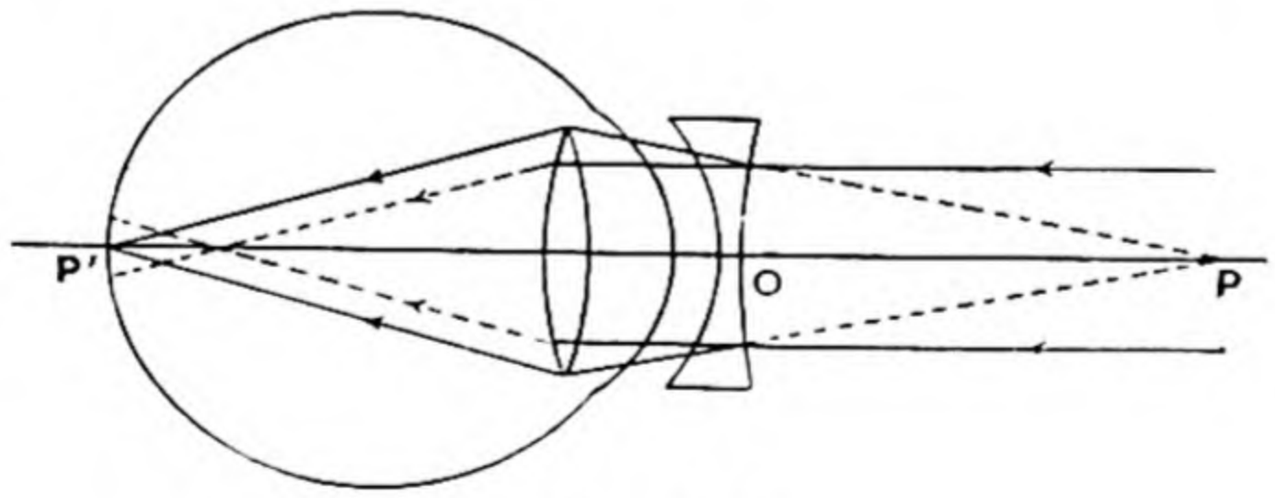

Fig. 101.

clearly because as an object approaches the eye its image recedes behind the crystalline lens nearer to the retina. This defect cannot be remedied by accommodation, because accommodation is only effected by increasing the curvatures of the surfaces of the crystalline lens and therefore making it still more convergent. The defect may be remedied by the use of a divergent lens. This makes rays from a distant object diverge before entering the eye and therefore the image is produced further from the front of the

eye, and by using a lens of the right power images are produced on the retina.

Let P be the "far point" of the eye, that is, the most distant point the rays from which will be brought to a focus at P' on the retina when the eye is unaccommodated. Then a lens that will enable the eye to see distant objects clearly must be such that it will cause a pencil of parallel rays to diverge from P and so be brought to a focus at P'.

If such a lens be placed at O, it is clear from the figure that OP is its focal length: and hence if the distance of the lens from the eye be increased, since P is a fixed point in relation to the eye, a lens of shorter focal length or greater power will be needed. The lens required is clearly divergent.

(2) **Hypermetropia or Long Sight.** This means that the eye cannot see near objects clearly, because the eyeball is too short and images of near objects are formed behind the retina. In other words the eye is not sufficiently convergent, or its power is too weak. Accommodation diminishes this defect somewhat but is insufficient and a convergent lens must be used to increase the power of the eye, or shorten the focal length of the combination, or as Newton expressed it "to supply the defect of plumpness in the eye."

Of course if an object is moved further from the eye its image will approach the crystalline lens, that is move forwards from behind the retina on to the retina, so that without artificial aid an object is seen more clearly by moving it further from the eye.

In a hypermetropic eye, when unaccommodated, rays must be converging when they enter the eye if they are to be brought to a focus on the retina.

Suppose that P is the fixed point behind the retina to which rays must converge at incidence on the eye in order to be brought to a focus at P' on the retina. Then a lens that will enable the eye to see distant objects clearly without accommodation must be such that it will make parallel rays converge to P, so that then the eye may bring them to a focus at P'. If such a lens be placed

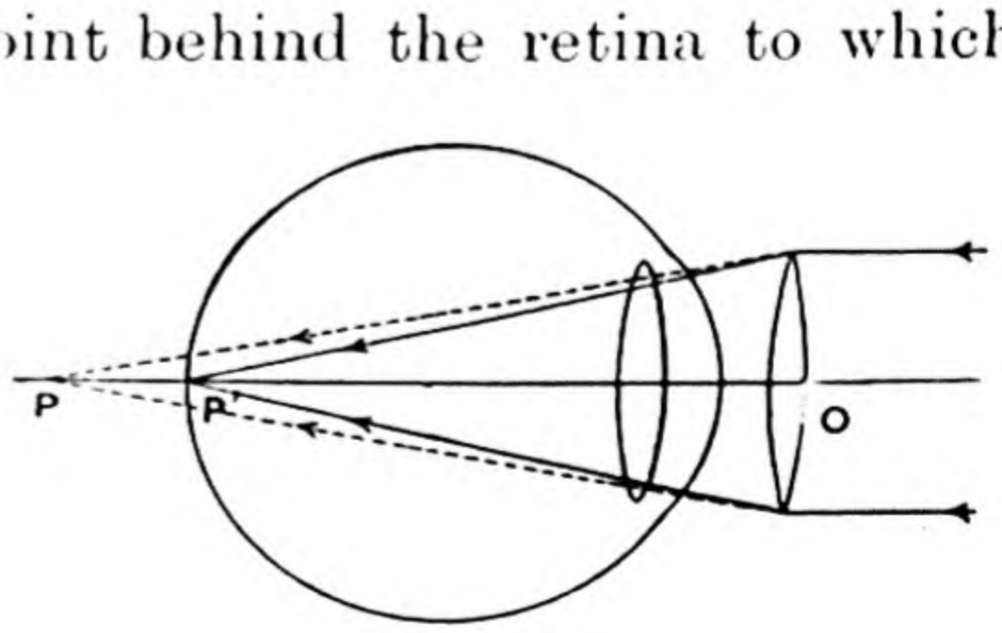

Fig. 102.

at O, then OP is clearly its focal length and it must be a convergent lens. When a suitable lens has been found for distant vision accommodation will enable the eye to see objects at lesser distances down to about 10 inches, provided the eye possesses a normal amount of accommodating power.

It is clear from the figure that if the lens is moved forwards from the eye so that OP is increased, a lens of longer focal length or smaller power will be sufficient for the required purpose. This explains why improved results can be obtained from spectacles which are too weak for the eyes that are to use them, by pushing them forwards down the nose.

(3) **Astigmatism.** If the eye were symmetrical about its axis, it would follow that when, say, a horizontal line at a given point was clearly visible, a vertical line or a line inclined at any angle to the horizontal at the same point would also be clearly visible. In many eyes this is not the case, the defect is known as astigmatism, and it is due to the want of symmetry of the eye about its axis, or more particularly to the fact that the horizontal and vertical curvatures of the cornea are not the same, a vertical section having greater or less curvature than a horizontal one. This defect can be remedied by the use of a cylindrical lens to add curvature in whatever direction it is required.

The defect of astigmatism may be combined with either of the other defects, and in that case spectacles are made with one surface cylindrical and the other spherical.

165. Vision through a lens. Simple microscope.

When a convergent lens is used as a simple microscope the object to be viewed is placed between the first principal focus and the lens so that a magnified, erect, virtual image is formed.

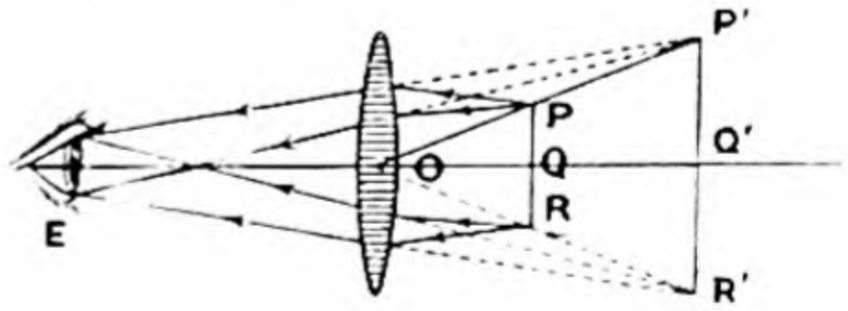

Fig. 103.

When an object is viewed directly its apparent magnitude depends on the angle it subtends at the eye and this is increased as the object is brought nearer to the eye, but it must not be brought nearer than the least distance of distinct vision, D say, and then its apparent magnitude is greatest.

When a magnifying glass is used placed *close to the eye* the object and image subtend practically the same angle POQ, or

$P'OQ'$ in the figure, and the **magnifying power** is measured by comparing the angle subtended by the image at the eye with the angle that the object would subtend if placed at the least distance of distinct vision.

Thus if we assume that the image is formed at the least distance of distinct vision, we have, for an eye close to the lens, $OQ' = D$, and if f be the numerical focal length,

$$\frac{1}{D} - \frac{1}{OQ} = -\frac{1}{f},$$

so that
$$OQ = \frac{fD}{f+D}.$$

Hence the *magnifying power*

$$= \frac{P'Q'}{OQ'} \div \frac{PQ}{D} = \frac{P'Q'}{PQ}$$

$$= \frac{OQ'}{OQ} = \frac{D(f+D)}{fD} = 1 + \frac{D}{f}.$$

166. Again to find the magnifying power when the same lens is held at a distance a from the eye and used to view an object at a distance u from the lens.

The magnifying power m

$$= \frac{P'Q'}{EQ'} \div \frac{PQ}{D}$$

$$= \frac{OQ'}{OQ} \cdot \frac{D}{a + OQ'},$$

also
$$OQ = u, \text{ and } \frac{1}{OQ'} - \frac{1}{u} = -\frac{1}{f},$$

so that
$$OQ' = \frac{uf}{f-u},$$

and
$$m = \frac{f}{f-u} \cdot \frac{D}{a + \frac{uf}{f-u}}$$

$$= \frac{fD}{af - au + fu}.$$

This is not the same thing of course as the angular magnification, which in this case

$$= \frac{P'Q'}{EQ'} \div \frac{PQ}{EQ} = \frac{OQ'}{OQ} \cdot \frac{EQ}{EQ'} = \frac{v(a+u)}{u(a+v)} = \frac{f(a+u)}{af - au + fu}.$$

167. Telescopes and Telescopic Systems.

A **telescope** is an arrangement of lenses on the same axis adapted for viewing distant objects.

For a normal sighted eye objects are viewed most easily by parallel rays, and when a telescope is so arranged that rays from a distant object emerge as parallel rays the instrument is said to be in *normal adjustment.*

A system of lenses on the same axis so adjusted that rays which are parallel at incidence emerge as parallel rays is called a **telescopic system of lenses.**

168. We shall now make some investigations which pertain to general systems and to telescopic systems in particular, and reserve the consideration of particular instruments for another chapter.

The condition for a telescopic system is that the power of the system be zero, or the focal length infinite.

Hence in the formula

$$v = \frac{Cu + D}{Au + B} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

we must have $A = 0$, and therefore from Arts. 80, 82 the principal foci and unit points are all at an infinite distance, and $BC = 1$. (Art. 79 (6).)

Hence, in this case $Bv - Cu - D = 0$; and if t be the distance between the first and last lenses there is a point which is its own conjugate, viz. that for which $v - u - t = 0$; and therefore

$$\frac{v}{Ct - D} = \frac{u}{Bt - D} = \frac{1}{C - B}.$$

This point is called the **centre** of the system.

If x, x' are the distances of a pair of conjugate points from the centre C,

$$x = u - \frac{Bt - D}{C - B} = \frac{Cu + D - B(u + t)}{C - B} = \frac{B(v - u - t)}{C - B},$$

and $$x' = v - \frac{Ct - D}{C - B} = \frac{C(v - t) + D - Bv}{C - B} = \frac{C(v - u - t)}{C - B}.$$

Therefore $$\frac{x'}{x} = \frac{C}{B} = \frac{1}{B^2};$$

and this ratio may be called the **elongation** of the telescopic system.

The linear magnification m is equal to $1/B$, and is independent of the position of the object.

If PQ, $P'Q'$ are an object and its image, the **magnifying power** M of such an instrument may be defined as the ratio of the angles subtended at C by $P'Q'$ and PQ, *i.e.*

$$M = \frac{P'Q'/x'}{PQ/x} = mx/x' = B.$$

The magnifying power as thus defined is therefore constant and independent of the position of the object.

169. Apparent Distance.

When an object is viewed through a combination of lenses its apparent distance* is the distance from the eye at which the object would have to be placed in order to subtend the same angle as the image subtends at the eye.

Fig. 104.

If PQ be the object, $P'Q'$ the final image, O, O' the first and last lenses of the combination and E the eye; we have

$$\frac{PQ}{\text{apparent distance}} = \frac{P'Q'}{EQ'}.$$

Therefore

$$\text{apparent distance} = \frac{PQ}{P'Q'}EQ' = \frac{EQ'}{m},$$

where m is the linear magnification.

In the notation of Art. 92, $OQ = u$, $O'Q' = v$,

$$\frac{1}{m} = Au + B, \text{ (Art. 93)} \quad \text{and} \quad v = \frac{Cu + D}{Au + B},$$

therefore

$$\begin{aligned}\text{apparent distance} &= (Au + B)(EO' + v)\\ &= x(Au + B) + Cu + D,\end{aligned}$$

where x is the distance of the eye behind the last lens of the combination.

* The term was introduced by Roger Cotes, first Plumian Professor at Cambridge. See R. Smith's *A Compleat System of Optics in four books*, Cambridge, 1738.

170. To shew that the apparent distance is unaltered if the eye and the object are interchanged.

The relation $$v = \frac{Cu + D}{Au + B}$$

is equivalent to $$-u = \frac{B(-v) + D}{A(-v) + C} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now imagine the usual diagram turned through two right angles so that the combination is reversed. This latter formula gives the distance $(-u)$ in front of the last lens of the image of an object which is at a distance $(-v)$ in front of the first lens, when the combination of lenses is reversed.

Hence if the eye be at a distance u behind the last lens of the reversed instrument and x be the distance of the object in front of the first lens, from (1), as in the last article, the

$$\text{apparent distance} = u(Ax + C) + Bx + D$$
$$= x(Au + B) + Cu + D.$$

This proves that the eye and the object may be interchanged without altering the apparent distance.

171. Magnifying power.

It is usual to take as the **magnifying power** of any instrument consisting of coaxial lenses what we have defined in Art. 77 as the angular magnification, that is, the ratio of the angles subtended at the eye by image and object. In general therefore the magnifying power depends both on the position of the eye and on that of the object in relation to the instrument.

In terms of apparent distance the magnifying power

$$= \frac{\text{actual distance of object}}{\text{apparent distance}}.$$

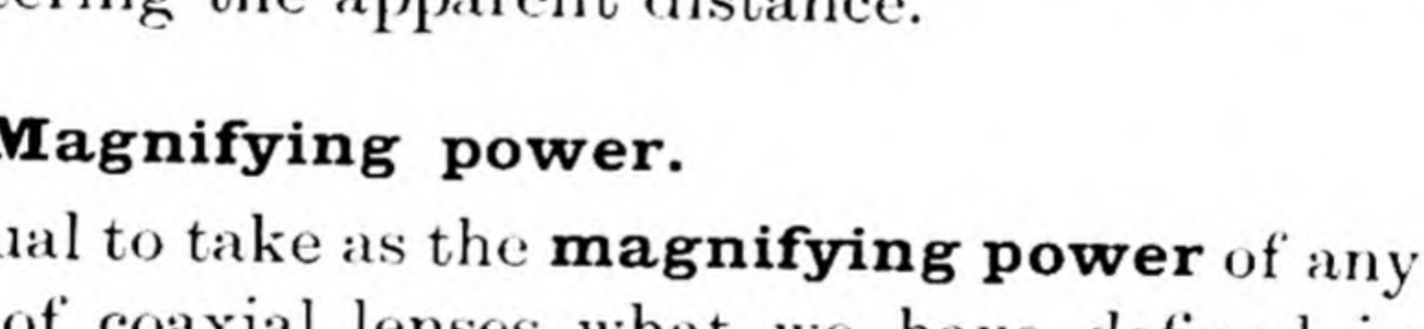

Fig. 105.

For if PQ, $P'Q'$ be the object and image and E the eye, and PP_1 parallel to QE meets EP' in P_1 and P_1Q_1 is perpendicular to

EQ, then $P_1Q_1 = PQ$ and EQ_1 is the apparent distance. Also

$$\text{magnifying power} = \frac{\text{angle } P'EQ'}{\text{angle } PEQ} = \frac{P_1Q_1/EQ_1}{PQ/EQ} = \frac{EQ}{EQ_1},$$

$$= \frac{\text{actual distance}}{\text{apparent distance}}.$$

172. When an instrument is used for viewing a very distant object, it is immaterial whether the distance of the object be measured from the eye or from one of the lenses or from any point at a finite distance from the instrument, and the angle subtended by the object at the eye will be practically the same as that which it subtends at any point at a finite distance from the eye.

Calling the lens nearest to the object the **object-glass** we can now shew that *the magnifying power of a telescope used for viewing a distant object is the ratio of the breadth at the object-glass to the breadth at the eye of a pencil of rays diverging from a distant point of the axis.*

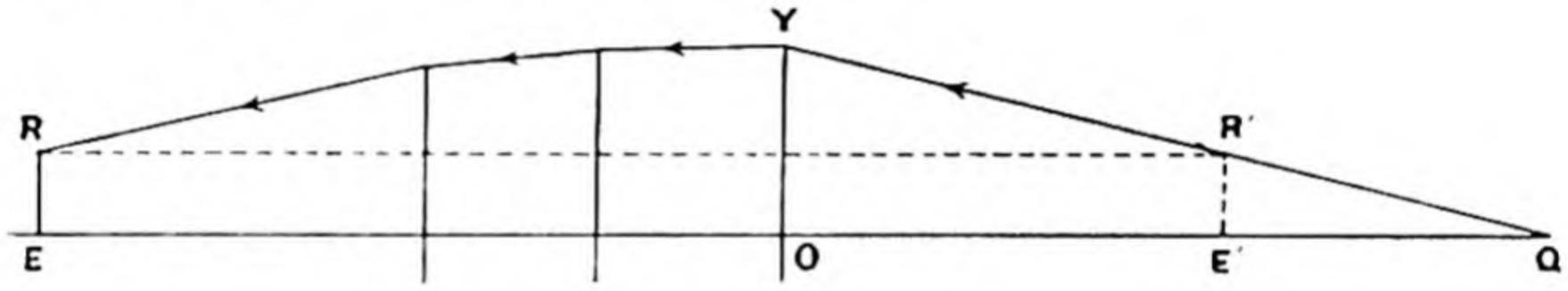

Fig. 106.

For if the pencil diverge from Q and RE be its breadth at the eye E and YO its breadth at the object-glass O, and we draw RR' parallel to the axis meeting QY in R' and draw $R'E'$ at right angles to the axis, then $E'Q$ would be the apparent distance of an object at E seen from Q, and $E'Q$ is therefore the apparent distance of an object at Q seen from E. Hence

$$\text{the magnifying power} = \frac{EQ}{E'Q}$$

or, since the distances are great,

$$= \frac{OQ}{E'Q} = \frac{OY}{E'R'} = \frac{OY}{ER},$$

as was to be proved.

173. Other expressions for magnifying power.

Again let E be the eye and E' its conjugate point with regard to the instrument. Provided E is not at a focus of the instrument, E' is at a finite distance.

Let an extreme ray from a distant object enter the eye E at an inclination α to the axis, the same ray must at incidence cross the axis at E' at an inclination α', say. Then we may take α' and

Fig. 107.

α to represent the angular magnitudes of the object and image, so that if m is the magnifying power

$$m = \frac{\alpha}{\alpha'},$$

and by Helmholtz's theorem this is equal to

$$\frac{\text{linear dimensions of an object at } E'}{\text{,, ,, ,, its image at } E} \quad \ldots\ldots\ldots\ldots (1).$$

This expression takes a variety of forms; thus, if F_2 is the second focus of the instrument supposed *not to be in normal adjustment*, the image of a distant object is formed at F_2 and we may therefore take EF_2 to represent the least distance of distinct vision D.

In this case m being the reciprocal of the linear magnification of an object at E' is equal to F/EF_2 or F/D, where F is the focal length of the *instrument*.

174. Eye-ring and entrance pupil.

Again all rays which enter through the object-glass must emerge through the image of the object-glass formed by the rest of the instrument, if they emerge at all.

This image is called the **eye-ring**. If the eye be placed at the eye-ring and the pupil be as large as the eye-ring it will embrace all the rays that get through the instrument. The centre of the eye-ring is therefore the best position for the eye. *When the eye is at the eye-ring*, E' is at the object-glass, and from (1)

$$m = \frac{\text{radius of object-glass}}{\text{radius of eye-ring}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

whether the instrument be in normal adjustment or not. This also follows as a corollary from Art. 172.

Again the lens nearest to the eye is called the **eye-glass**, and it is clear that all rays which emerge through the eye-glass must

at incidence fall within the image of the eye-glass formed by the rest of the instrument. This image is called the **entrance pupil.** *If the eye be placed at the eye-glass,* E' is at the entrance pupil and from (1)

$$m = \frac{\text{radius of entrance pupil}}{\text{radius of eye-glass}} \quad \ldots\ldots\ldots\ldots\ldots(3),$$

whether the instrument be in normal adjustment or not.

175. Thus we see that the magnifying power in general depends on the position of the eye; and this is always the case *except when the instrument is in normal adjustment,* for then the emergent rays are parallel and the inclination of any one of them to the axis may be taken as the angle which the image subtends at the eye, so that the eye may be moved along the axis without altering the magnifying power, and in this case expressions (2) and (3) both represent the magnifying power irrespective of the position of the eye.

176. If a telescope in normal adjustment consist of two groups of lenses of focal lengths F, f the magnifying power is $-F/f$.

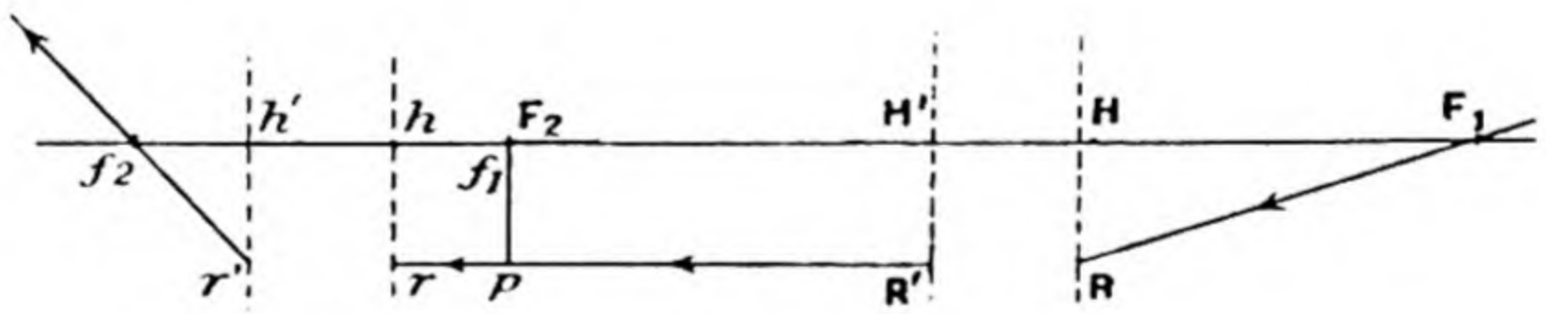

Fig. 108.

Represent the two parts of the instrument by their cardinal points. In normal adjustment the second focus of the first group of lenses must coincide with the first focus of the second group. Suppose rays from a distant object to form an image pF_2 in the common focal plane, and let F_1R, $R'pr$, $r'f_2$ be parts of the path of a ray which is parallel to the axis between the unit planes. Then the inclinations of F_1R and $r'f_2$ to the axis may be taken as the angular magnitudes of the object and image, so that

$$m = -\frac{r'f_2h'}{RF_1H} = -\frac{F}{f}.$$

EXAMPLES.

1. The first and second Principal Foci of the normal eye are respectively 13·75 millimetres in front of and 22·83 mm. behind the cornea while the first and second Principal Points are respectively 1·75 mm. and 2·12 mm. behind the cornea : find the positions of the Nodal Points. (Coll. Exam. 1906.)

2. A thin convergent lens of focal length 6 inches is used to view an object 4 inches from the lens. Where must an eye be placed in order that the angular magnification may be 2? (Coll. Exam. 1911.)

3. On the assumption that a normal-sighted person sees clearly at all distances greater than 12 inches, shew that, if he wears convex glasses of focal length $10\frac{3}{4}$ inches at a distance half an inch from his eyes, he will be able to see clearly at distances between 6 and $11\frac{1}{4}$ inches only.

(Coll. Exam. 1912.)

4. A certain person cannot see objects clearly at a less distance than 8 ins. from his eye, or at a greater distance than 2 ft. Find within what limits of distance from his eye a concave mirror of radius 1 ft. may be placed in order that he may see clearly the reflection of his eye in the mirror.

(St John's Coll. 1913.)

5. A long-sighted eye, when unaccommodated, is such that a pencil of rays converging to a point distant p cm. behind the front of the eye is brought to a focus on the retina: determine the focal length of the lens which placed close to the eye will enable it to see without accommodation an object placed q cm. from the eye front. In what case will a lens slightly too weak in power be made effective by slight advance from the eye?

(M. T. i. 1909.)

6. A person's vision is such that for rays to be brought to a focus on the retina without accommodation, they must converge towards a point 30 cms. behind the front of the eye. Find the focal length of a lens placed close to the eye that will render an object 40 cms. away clearly visible. If the lens be moved a small distance from the eye, how must the object be moved in order still to be visible clearly? (M. T. ii. 1911.)

7. A person finds that to see a finely divided scale most distinctly he must place it a foot distant from his eye : he also finds that when he looks at a similar scale through a convex lens held close to his eye, and sees it most distinctly, the graduations look 7 times as large as in the first case. Shew that the focal length of the lens must be 2 inches. (Coll. Exam. 1891.)

8. Explain the method by which the eye adapts itself to distinct vision of objects at different distances.

Examine whether, in view of this adjustment, the apparent sizes of objects at different distances, as seen by the naked eye, give an entirely correct measure of the angles they subtend at the observer.

(St John's Coll. 1899.)

9. A candle is placed in front of a thin double-convex lens whose surfaces are of equal radii. An eye on the same side of the lens as the candle sees two images, one always erect and a second erect or inverted according to the distance of the lens from the candle. Explain the origin of these images and determine their positions for a given position of the candle. (Coll. Exam. 1891.)

10. If an instrument be such that an observer can place his eye at a principal focus, then the apparent distance of all objects so viewed will be the same. (Coll. Exam. 1904.)

11. Shew that when an object is viewed through a lens of thickness t and refractive index μ, the apparent distance of the object is less than its true distance by

$$\frac{\mu-1}{\mu}t-\frac{t}{\mu}(\kappa_1 u+\kappa_2 v)-K u v,$$

where u, v are the distances of the object and the eye from the lens, K is the power of the lens and κ_1, κ_2 the powers of its surfaces. (M. T. 1903.)

12. Shew that optical systems exist for which the magnifications of the images of all objects, wherever situated on the axis, are the same.

Shew that by such a system the angle which the object subtends at a certain point on the axis is increased in the same ratio as the actual magnitude of its image is diminished. (St John's Coll. 1896.)

13. Shew that two equal hemispheres, of the same kind of glass, form a telescopic system if their adjacent plane faces are at a distance apart equal to $2/\mu(\mu-1)$ of the radius of either. (St John's Coll. 1899.)

14. If three thin convex lenses of focal lengths f_1, f_2, f_3 are placed on the same axis, the distances between them being a_3, a_1, prove that they form a telescopic system provided

$$\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}-\frac{a_1}{f_2 f_3}-\frac{a_2}{f_3 f_1}-\frac{a_3}{f_1 f_2}+\frac{a_1 a_3}{f_1 f_2 f_3}=0,$$

where $a_2=a_3+a_1$.

Shew also that the angular magnification is

$$1-\frac{a_2}{f_3}-\frac{a_3}{f_2}+\frac{a_3 a_1}{f_2 f_3},$$

and that the centre divides the distance between the extreme lenses in the ratio

$$\frac{f_1}{f_3}\cdot\frac{a_1 a_2 a_3-a_2^2 f_2-a_3^2 f_3}{a_1 a_2 a_3-a_2^2 f_2-a_1^2 f_1}.$$ (Coll. Exam. 1899.)

CHAPTER XII

OPTICAL INSTRUMENTS

177. The Common Astronomical Telescope.

This instrument consists of a convergent object-glass and eye-piece; in its simplest form the eye-piece consists of a single convergent lens. Let F, f be the numerical focal lengths of the object-glass O and the eye-glass E.

Let F_1, F_2 and f_1, f_2 be their principal foci.

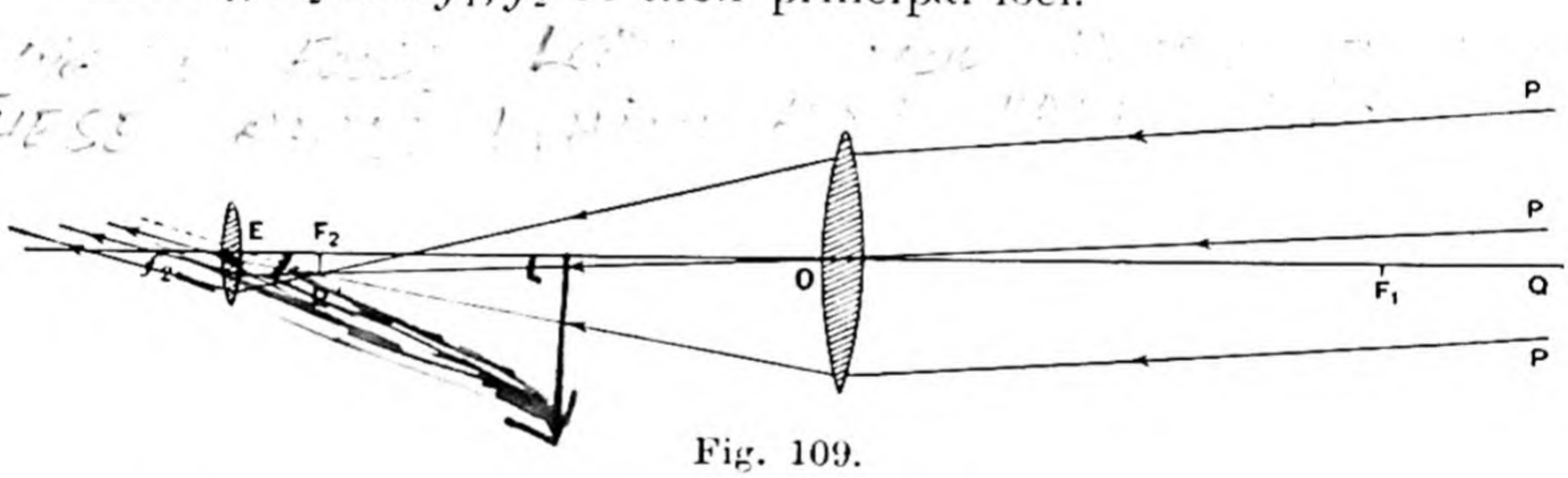

Fig. 109.

Rays from a distant object are brought to a focus in the second focal plane of the object-glass, forming a real image in this plane through F_2. The eye-piece is used to magnify this image. In *normal adjustment* rays which are parallel at incidence must emerge parallel, so that the image formed by the object-glass must also be in the first focal plane of the eye-glass. That is, in normal adjustment f_1 coincides with F_2.

Let rays from a distant point P be brought to a focus at P' in the common focal plane. The rays are parallel at incidence and that one which goes through the centre O of the object-glass undergoes no deviation. After striking the eye-glass, since the rays are proceeding from a point P' on its first focal plane, they emerge parallel to one another and their direction is parallel to $P'E$, where E is the centre of the eye-glass, because a ray $P'E$ would undergo no deviation.

178. The **magnifying power**, being the ratio of the angles subtended by the object and image at the eye, may be measured by the ratio of the inclinations of the parallel pencils to the axis. For if the axis meets the object in Q, PQ would subtend at the eye an angle practically equal to POQ, and the angle subtended by the image of PQ would be the inclination of any emergent ray to the axis. Hence for any convenient position of the eye the magnifying power

$$= \frac{\text{angle } P'EF_2}{\text{angle } P'OF_2} = -\frac{OF_2}{EF_2} = -\frac{F}{f},$$

the minus sign being used to indicate that the image is ~~an inverted one~~. virtual & erect

This result also follows directly from Art. 176.

179. Field of view.

In any telescope rays which enter the object-glass do not all emerge through the eye-glass. In order to emerge from the eye-glass a ray must traverse all intermediate lenses and get through the apertures of all "stops," a stop being an opaque diaphragm with a circular aperture inserted in order to confine the emergent rays to rays near the axis. An emergent ray will therefore have to be directed at incidence on the object-glass so as to pass through the image of every lens and stop formed by all that part of the instrument from the particular lens or stop up to the object-glass. These images limit the field of view and there will be one which limits the field of view more than all the rest, and it must be found and utilized for determining the angular radius of the field of view in the manner shewn below.

180. For the common astronomical telescope with a single eye-glass, the field of view is limited by the *entrance pupil* or image of the eye-glass formed by the object-glass. (Art. 174.)

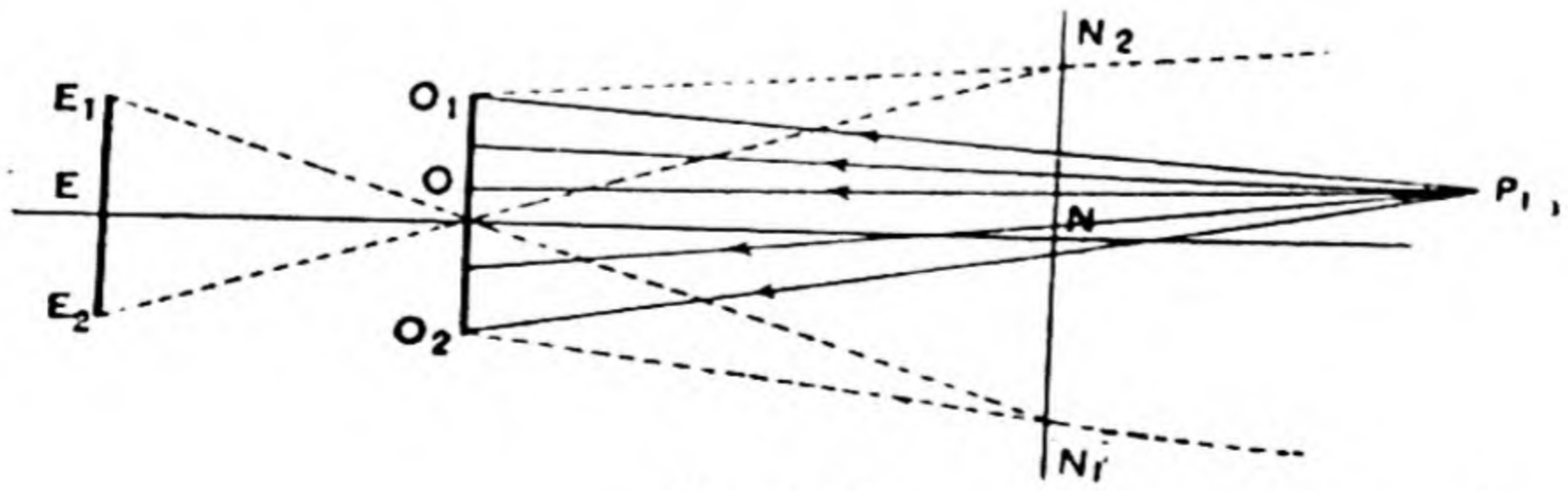

Fig. 110.

Let N_1NN_2 be the image of the eye-glass E_1EE_2 formed by the object-glass O_1OO_2. Then any source of light in front of the object-glass will emit rays that fill the object-glass, but only those rays will emerge through the eye-glass that cross the entrance pupil N_1NN_2 at incidence. Thus any point P_1 between the lines O_1N_2, O_2N_1 emits a pencil which fills the object-glass and the whole of which falls within N_1N_2. Hence any point between O_1N_2, O_2N_1 will be seen by a pencil filling the object-glass.

Again, considering a point P_2 outside the line O_1N_2 but within the angle formed by O_1N_1 and O_2N_2, only part of the pencil from P_2 that falls on the object-glass falls within N_2N_1 and so the point P_2 would be visible through the telescope but less brightly than

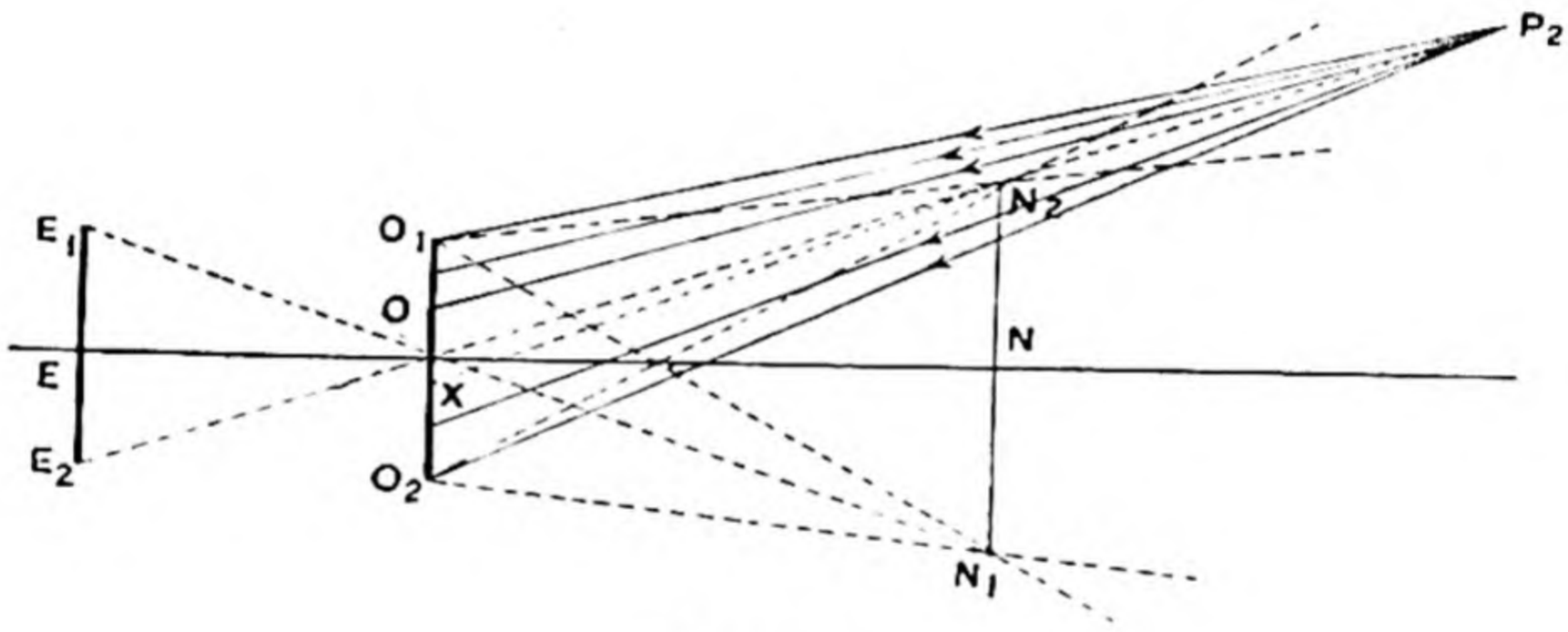

Fig. 111.

if it were within the former limits, for it is clear from the figure that only those rays from P_2 that fall in the limited area O_2X of the object-glass will emerge from the eye-glass, P_2N_2X being a straight line. In other words, P_2 will be visible but not by a full pencil.

In like manner it can be seen that points without the cone of revolution of which O_1N_1 and O_2N_2 are generators will not be visible at all.

Hence for a very distant object to be seen by a full pencil the parallel rays must not make with the axis a greater angle than the inclination of O_1N_2. This angle is called the **angular radius of the field of view** for full pencils and is denoted by Θ. Thus we have

$$\Theta = \frac{N_2N - O_1O}{ON}.$$

and if y_o, y_e are the radii of the object-glass and eye-glass, we have

$$N_2N = E_2E\,\frac{F}{F_2E} = \frac{F}{f}\,y_e$$

and
$$ON = F + F_1N = F + \frac{F^2}{EF_2} = F + \frac{F^2}{f},$$

therefore
$$\Theta = \frac{Fy_e - fy_o}{F(F+f)}.$$

Similarly the angular radius Θ_1 of the whole visible field of view is the inclination of N_2O_2 to the axis, so that

$$\Theta_1 = \frac{N_2N + OO_2}{ON} = \frac{Fy_e + fy_o}{F(F+f)}.$$

Cor. $\dfrac{\text{radius of entrance pupil}}{\text{radius of eye-glass}} = \dfrac{NN_2}{EE_2}$

$$= -\frac{F}{f} = \text{magnifying power},$$

as in Art. 174.

181. Radius of stop.

It is usual to place a stop in the common focal plane to intercept the "ragged edge" of the field of view, that is the part that is not seen by full pencils. To determine the radius of the stop we observe that if parallel rays making an angle Θ with the axis fall on the object-glass, they will converge to a point P' on the rim of the stop.

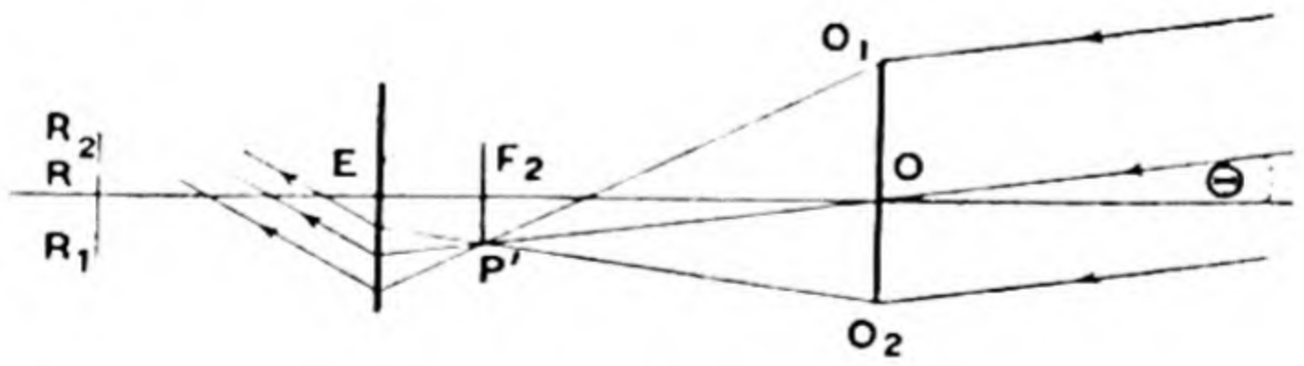

Fig. 112.

Hence the radius required is

$$P'F_2 = F_2O\,.\,\Theta = F\,.\,\Theta = \frac{Fy_e - fy_o}{F+f}.$$

182. The eye-ring.

If the eye be placed at the image of the object-glass formed by the eye-glass, it will clearly receive all rays incident on the object-glass that emerge through the eye-glass.

If R_1RR_2 be the eye-ring, since it is the image of O_1OO_2 formed by the eye-lens, we have

$$f_2R \,.\, f_1O = -f^2 \text{ and } R_1R/O_1O = -f/F_1O,$$

so that

$$RE = f + f^2/F,$$

and

$$\frac{\text{radius of object-glass}}{\text{radius of eye-ring}} = -\frac{F}{f} = \text{magnifying power},$$

as in Art. 174.

183. In telescopes used for astronomical purposes there is generally an arrangement of crossed spider lines fitted in the second focal plane of the object-glass where the real image is formed. The cross-wires, as they are called, and the image are then viewed together through the eye-piece and the angular distances of distant points can be measured by means of the cross-wires, one of which can be adjusted by a micrometer.

184. Adjustment of the eye-piece.

When a telescope is to be used by a person who sees most clearly at a distance D, the eye-piece must be adjusted so as to produce a final image at a distance D from the eye.

Let x be the distance that the eye-piece is pushed in to achieve this result, then with our former notation $F_2f_1 = x$.

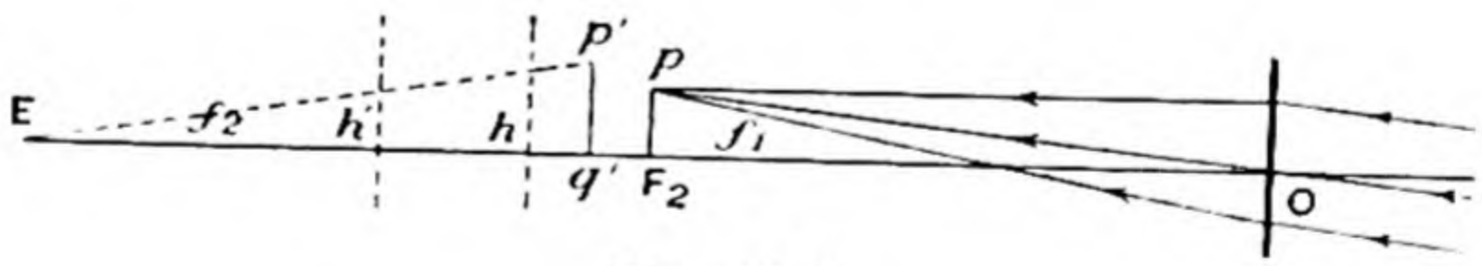

Fig. 113.

The object-glass forms a real image pF_2, and the eye-piece—represented diagrammatically by its cardinal points f_1, f_2, h, h'—produces an image $p'q'$ such that $Eq' = D$, where E is the eye.

We have $f_2q' = f^2/x$, and there are two cases to consider according as (1) the eye is at the eye-ring in the adjusted position of the eye-piece or (2) at the same distance from the eye-glass as the eye-ring would be in normal adjustment.

In the first case

$$Ef_2 = f^2/f_1O = f^2/(F - x).$$

Therefore

$$D = Ef_2 + f_2q' = f^2/(F - x) + f^2/x;$$

or

$$x^2 - xF + f^2F/D = 0.$$

This gives

$$x = \tfrac{1}{2}F \pm \tfrac{1}{2}\{F^2 - 4f^2F/D\}^{\frac{1}{2}};$$

and the minus sign is clearly the one to be taken, because in normal adjustment $D = \infty$ and $x = 0$.

In the second case

$$Ef_2 = f^2/F.$$

Therefore

$$D = Ef_2 + f_2q' = f^2/F + f^2/x,$$

and

$$x = Ff^2/(FD - f^2).$$

The magnifying power is minus the ratio of the angles $p'Eq'$ and pOF_2, *i.e.*

$$-\frac{p'q'}{D} \times \frac{F}{pF_2} = -\frac{F}{D}\cdot\frac{p'q'}{pF_2} = -\frac{F}{D}\frac{f}{x}.$$

In the first case this

$$= \frac{-2Ff}{D\{F - (F^2 - 4f^2F/D)^{\frac{1}{2}}\}}$$

$$= -\{F + (F^2 - 4f^2F/D)^{\frac{1}{2}}\}/2f.$$

And in the second case the value is

$$-(FD - f^2)/fD \quad \text{or} \quad -(F/f - f/D).$$

185. Examples. 1. Draw a careful diagram of an astronomical telescope, in which the aperture of the object-glass is 4 cm. and its focal length 22·5 cm., the aperture of the eye-glass 1·2 cm. and its focal length 1·5 cm. Trace the course of a pencil of rays from a distant object as viewed by an eye adapted to see objects by parallel rays; and obtain the radius of the stop which will exclude all but whole pencils.

Find what distances the eye-glass must be moved (i) to view by parallel rays an object at a distance of 45 metres, (ii) to view a distant object when the observer's distance of distinct vision is 30 cm., the eye being at the eye-glass. (Coll. Exam. 1908.)

2. A telescope is to be 12 inches long and to magnify eight diameters; determine the focal lengths of the two component lenses.

(St John's Coll. 1910.)

3. If the focal lengths of the glasses of an astronomical telescope be 1 inch and 1 foot and if the telescope be adjusted to view a star, shew that an object at a distance of 7 feet from the object-glass can be distinctly seen by a normal eye placed close to a lens whose first principal focus is distant $\frac{1}{2}$ an inch from the eye-glass. (Coll. Exam. 1913.)

4. Prove that the magnifying power of a telescope adjusted for distinct vision at a distance Δ is f/Δ, where f is the focal length of the instrument.

(St John's Coll. 1898.)

5. The eye-glass and object-glass of an astronomical telescope are of focal lengths 9 in. and 36 in. respectively. Find the magnifying power when the eye is placed at the eye-point and the eye sees most distinctly at a distance of 9 in.

(Coll. Exam. 1911.)

6. Find the magnifying power of an astronomical telescope for an observer, whose eye is at a distance a from the eye-lens, and whose distance of distinct vision is Δ.

(Coll. Exam. 1906.)

7. If an astronomical telescope is adjusted for very distant objects, and m is its magnifying power, prove that the distances x, y of a near object and its image from the common focal plane of the object-glass and eye-glass are connected by the relation $x+m^2y=2ml$, where l is the length of the telescope.

(Coll. Exam. 1902.)

8. Find the magnifying power of a simple astronomical telescope in normal adjustment and shew that, if the eye-lens be pushed in a distance x and the eye be supposed placed at the image of the object-lens in the eye-lens, the magnifying power is diminished by x/F of its previous value, where F is the focal length of the object-glass.

(Coll. Exam. 1907.)

186. Defects of object-glasses and eye-pieces.

Our previous investigations have mainly been based on the hypothesis that the rays considered make with the axis angles so small that the squares of their circular measures are negligible. In the use of a telescope such as we have described this hypothesis does not hold good, and defects in the image arise from the fact that rays diverging from a point do not all pass through the same point after traversing a lens, if the squares or higher powers of the angles of divergence cannot be neglected. These defects take the form of blurring and curvature of the image and are generally classed under defects due to spherical aberration.

Another defect is chromatic aberration.

187. Eye-pieces.

An astronomical telescope with a single lens for eye-piece is of no use for ordinary terrestrial purposes on account of the inverted image that it forms. If however a convergent lens be placed between the object-glass and eye-glass, it is possible to make the image erect.

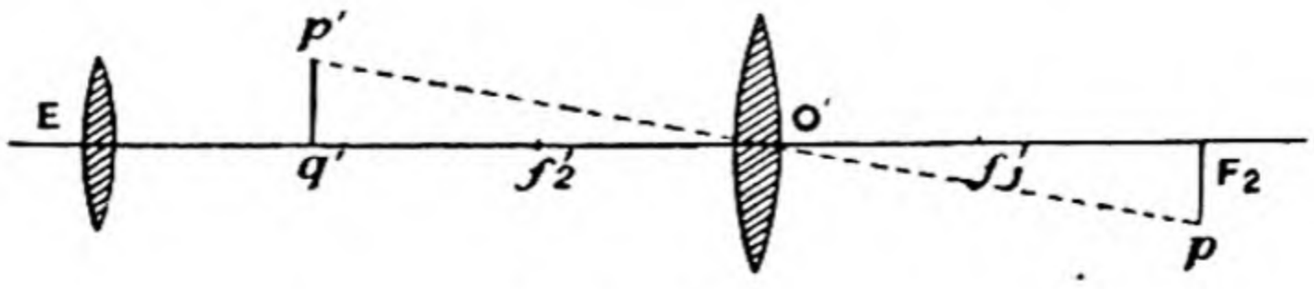

Fig. 114.

Thus if a lens O' (Fig. 114) of focal length f' be inserted so that the real inverted image pF_2 formed by the object-glass is at a distance $2f''$ from this lens, this lens will produce an equal erect image $p'q'$ at a distance $2f''$ on the opposite side, and this can then be viewed through a single lens E or a compound eye-piece.

In order to obtain a larger image two lenses are sometimes used in place of O', the first is so placed that pF_2 lies between the lens and its first focus, so that it forms an enlarged but still inverted virtual image $p'q'$: the second lens O' is then placed at a distance twice its focal length from $p'q'$ and produces an equal erect image $p''q''$ on its other side as before.

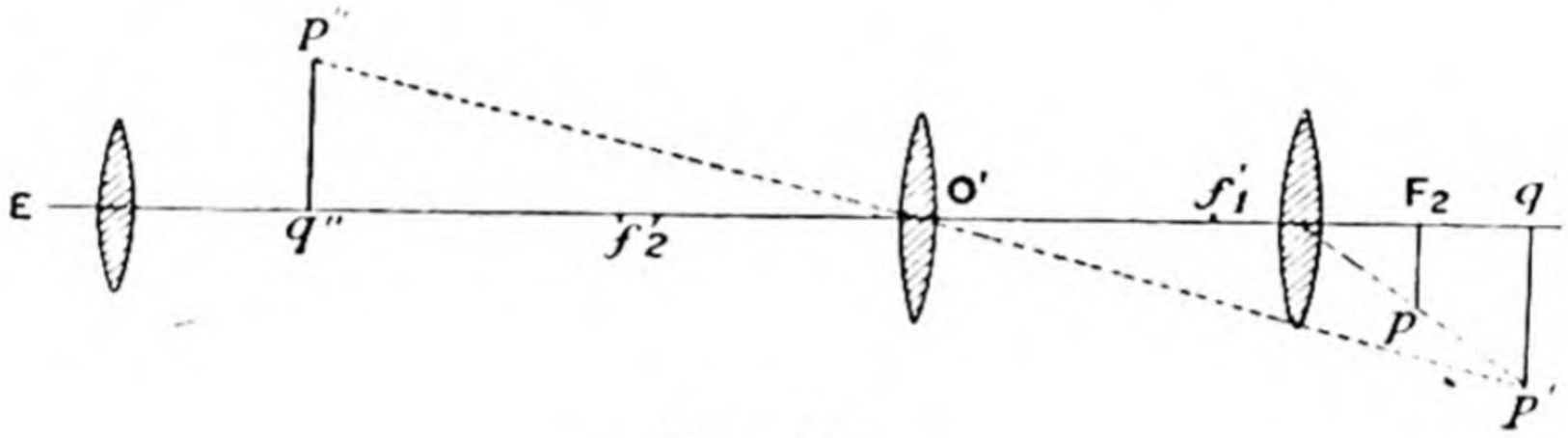

Fig. 115.

188. In general the advantages of using a compound eye-piece are that the field of view may be increased, and that chromatic aberration may be eliminated.

Ramsden's eye-piece consists of two convergent lenses of the same focal length f. The condition for achromatism (see Art. 137) would require the distance between the lenses to be f, but the principal focal planes of the combination would then coincide with the lenses and it would be impossible to adjust the cross-wires (Art. 183) in the common focal plane of the object-glass and eye-piece. As in other respects, this eye-piece is well suited for astronomical purposes, the distance between the lenses is fixed at $\frac{2}{3}f$, they are plano-convex with their plane faces outwards.

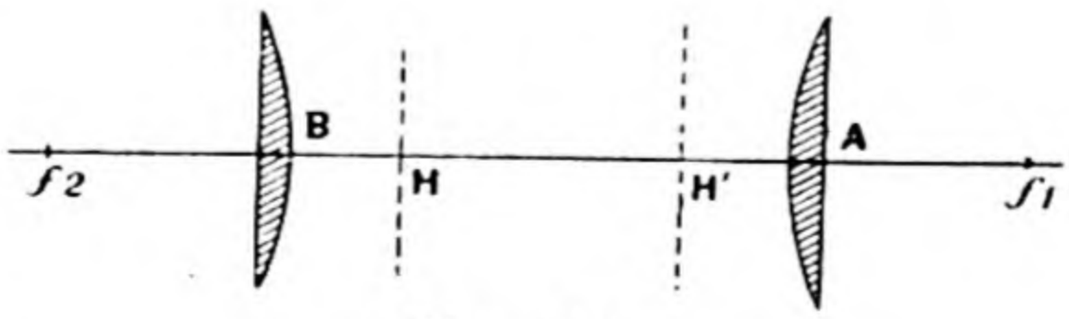

Fig. 116.

The principal foci can easily be shewn to be at a distance $\frac{1}{4}f$ in front of the field-lens A and behind the eye-lens B, and the unit planes lie between the lenses at distances $\frac{1}{2}f$ from the field-lens

and eye-lens respectively. In normal adjustment the object-glass will produce a real image at f_1.

A pencil of rays from a distant object may however be traced through a telescope fitted with a Ramsden's eye-piece, without the determination of the cardinal points of the instrument. Thus in normal adjustment the image $p'q'$ formed by the field-glass must be at a distance f in front of the eye-glass. But $p'q'$ is the image formed by the field-glass of the real image pF_2 formed by the object-glass at its second focus; therefore

$$\frac{1}{Aq'} - \frac{1}{AF_2} = -\frac{1}{f},$$

and

$$Aq' = \tfrac{1}{3}f,$$

so that

$$AF_2 = \tfrac{1}{4}f.$$

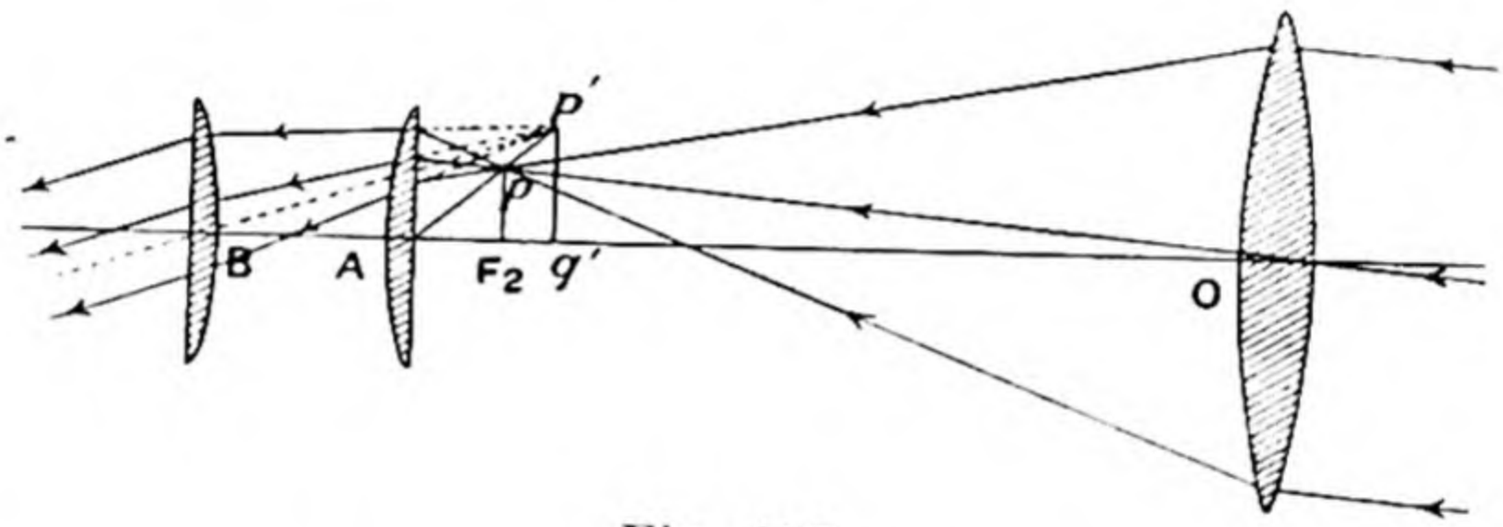

Fig. 117.

Hence rays which are parallel at incidence on the object-glass are brought to a focus at p and after passing through the field-glass diverge from p', the image of p, and then emerge from the eye-glass parallel to $p'B$.

The magnifying power in normal adjustment is from Art. 176 the negative ratio of the focal length of the object-glass to that of the eye-piece or $-F/\frac{3}{4}f$; or, we have

$$m = -\frac{p'Bq'}{pOF_2} = -\frac{p'q'}{pF_2}\cdot\frac{F_2O}{Bq'} = -\frac{Aq'}{AF_2}\cdot\frac{F}{f} = -\frac{4}{3}\frac{F}{f}.$$

The adjustment of Art. 184 for a final image at a given distance from the eye holds in this case.

189. Huyghens' eye-piece.

This eye-piece consists of two convergent lenses of focal lengths $3f$ and f at a distance $2f$ apart. The lenses are plano-convex with both plane faces turned to the eye.

The condition for achromatism is satisfied.

The field-lens is of course the one of larger focal length.

The first and second principal foci of the combination are found to be at a distance $\frac{1}{2}f$ in front of and behind the eye-glass, and the unit points are at a distance f behind and in front of the eye-glass, the focal length of the combination being $\frac{3}{2}f$.

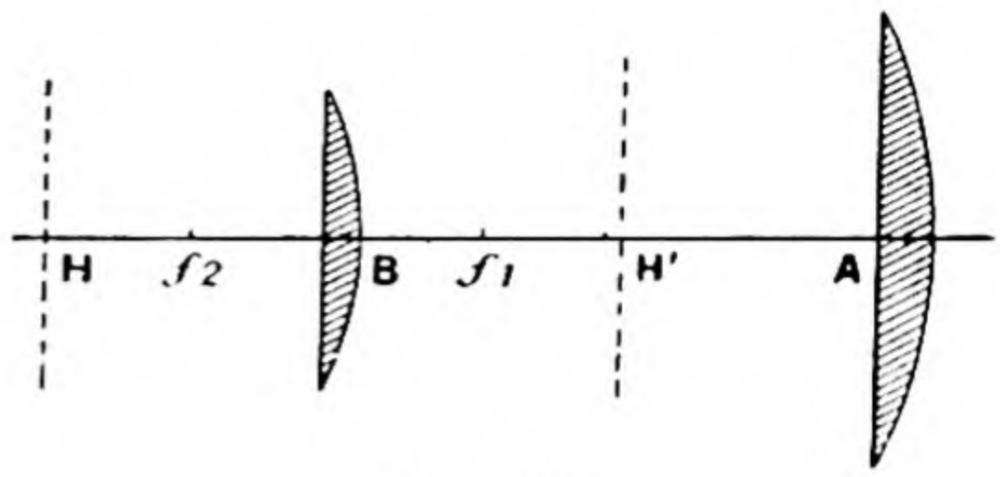

Fig. 118.

In normal adjustment the object-glass forms a real image at f_1, *i.e.* f_1 is also the second focus of the object-glass.

To trace a pencil of rays through a telescope fitted with a Huyghens' eye-piece, we observe that since the rays must emerge from the eye-glass as a parallel pencil, the field-glass must form an image $p'q'$ at the first focus of the eye-glass, *i.e.* we must have

$$Bq' = f.$$

Therefore, if pF_2 is the image formed by the object-glass,

$$\frac{1}{Aq'} - \frac{1}{AF_2} = -\frac{1}{3f}, \text{ and } Aq' = -f,$$

so that

$$AF_2 = -\tfrac{3}{2}f.$$

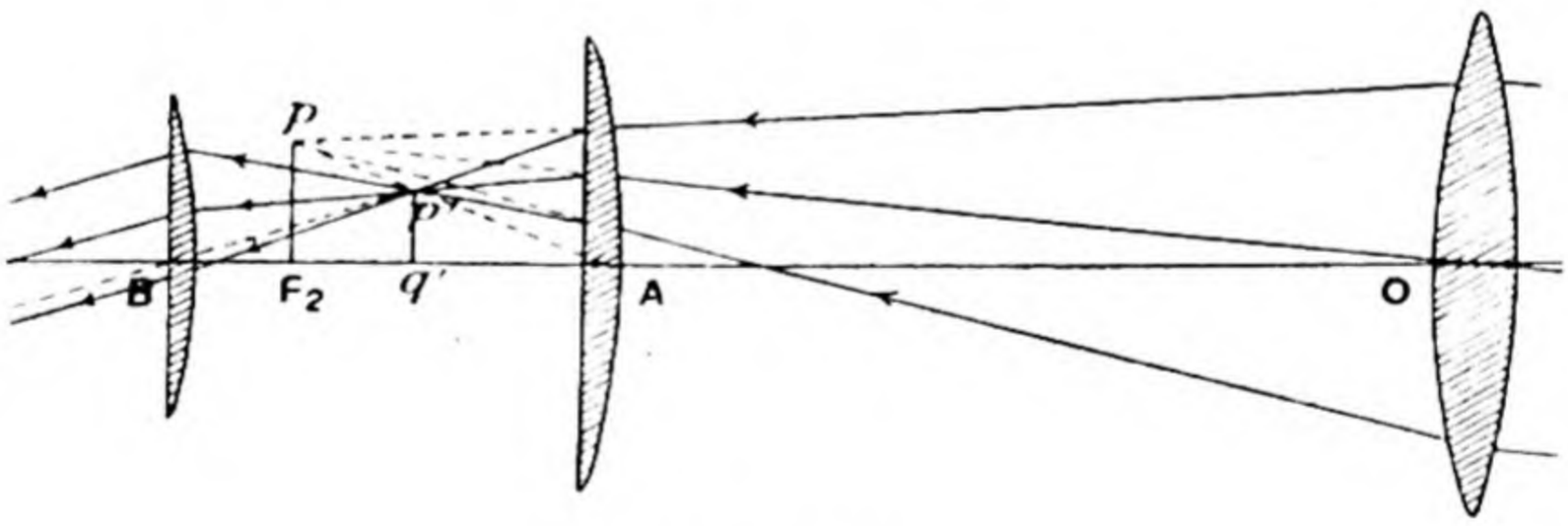

Fig. 119.

Hence rays which are parallel at incidence on the object-glass converge towards p but are intercepted by the field-glass and brought to a focus at p', the image of p formed by the field-glass, they then strike the eye-glass and emerge parallel to $p'B$.

The magnifying power in normal adjustment is, from Art. 176, the negative ratio of the focal length of the object-glass to that of the eye-piece, *i.e.* $-F/\frac{3}{2}f$; or

$$m = -\frac{p'Bq'}{pOF_2} = -\frac{p'q'}{pq}\cdot\frac{F_2O}{Bq'} = -\frac{Aq'}{AF_2}\cdot\frac{F}{f} = -\frac{2}{3}\frac{F}{f}.$$

The adjustment of Art. 184 for a final image at a given distance from the eye also holds in this case.

This eye-piece is not well adapted for astronomical purposes, because if cross-wires were placed in the plane of the image $p'q'$ which is viewed through the eye-glass, the wires would be viewed through the eye-lens only while the image would be seen by rays that have passed through both lenses of the eye-piece and therefore, except close to the centre of the field, the image and the wires would be differently distorted.

190. Examples. 1. Find how much longer a telescope will have to be which is fitted with a Ramsden's eye-piece, than one fitted with the single equivalent lens, when the telescope is directed to a distant object.

(Coll. Exam. 1908.)

2. Shew that the field of view of a telescope in normal adjustment fitted with a Ramsden's eye-piece, supposed limited by the field-lens of the eye-piece, is greater than that of the simple astronomical telescope formed by the object-glass and the equivalent lens even if the aperture of the equivalent lens be taken as great as that of either lens of the eye-piece.

(Coll. Exam. 1906.)

3. Shew that if F be the focal length of the object-glass of an astronomical telescope fitted with a Ramsden's eye-piece in which the lenses are of focal length f and at distance $\frac{2}{3}f$, the magnifying power when viewing a very distant object is $F(f+4d)/3df$, d being the distance of vision for which the eye is accommodated, and the eye being placed at the eye-lens. (St John's Coll. 1908.)

4. An astronomical telescope is fitted with a Huyghens' eye-piece of focal length ϕ; the semi-apertures of field-lens and eye-lens are a, b respectively, the object-glass has focal length F and semi-aperture c. Shew that the angular radius of the bright field is the lesser of the quantities

$$\frac{Fa-\phi c}{F(F-\phi)}, \qquad \frac{3(Fb-\phi c)}{F(F+3\phi)}.$$ (St John's Coll. 1906.)

5. In viewing a distant object through an astronomical telescope fitted with Huyghens' eye-piece, shew that for an eye placed close to the eye-lens to see the image at a distance λ the eye-piece must be pushed in a distance

$$9f^2/2(2\lambda+f),$$

f being the focal length of the eye-lens. (Coll. Exam. 1894.)

191. Galileo's Telescope.

This instrument consists of a convergent object-glass O and a divergent eye-glass E. In normal adjustment the second focus F_2 of the object-glass coincides with the first focus f_1 of the eye-glass, and the distance between the lenses is the difference between their numerical focal lengths $F-f$.

Parallel rays from a distant point p are made by the object-glass to converge towards a point p' in the common focal plane, and are intercepted by the eye-glass and made to emerge parallel to the

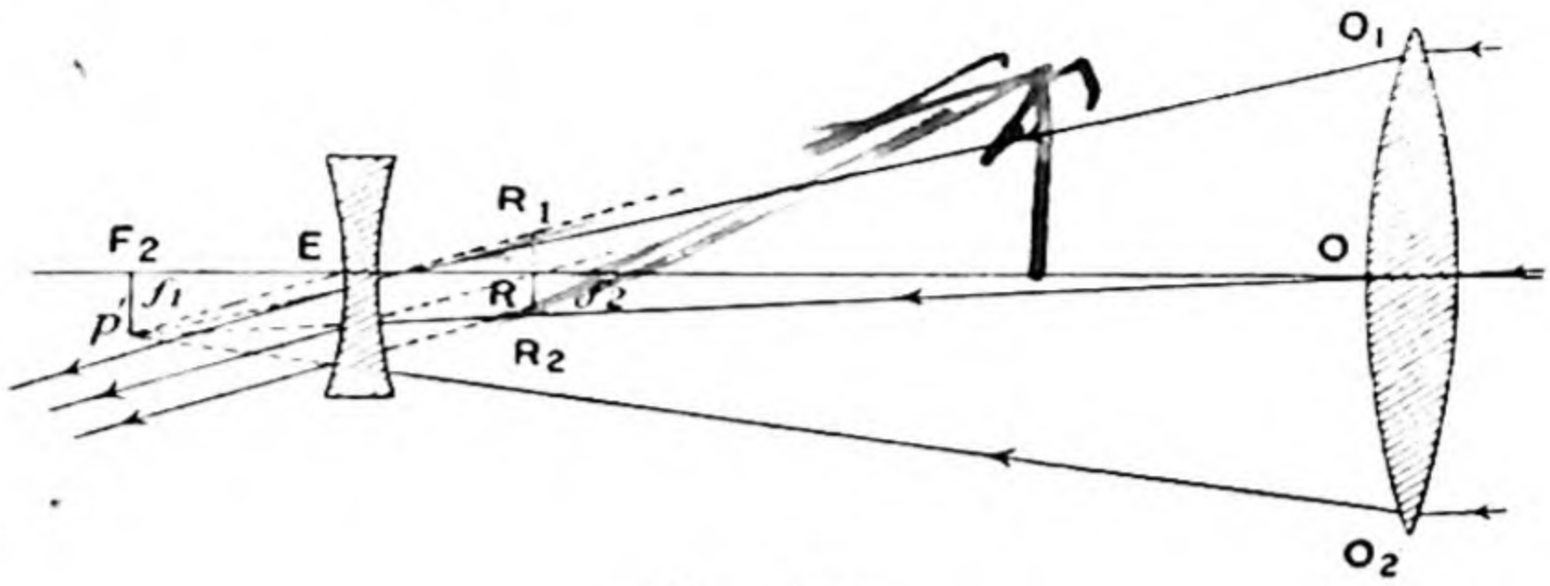

Fig. 120.

line Ep'. It is clear from the figure that the image is erect and magnified. The magnifying power is given by

$$m = \frac{p'EF_2}{p'OF_2} = \frac{F}{f} \quad \text{(see also Art. 176).}$$

The eye-ring is at R, where $f_2R \,.\, f_1O = -f^2$; hence the eye-ring falls between E and f_2, *i.e.* within the instrument.

If the emergent parallel pencil is produced backwards to the plane of the eye-ring, it cuts it in a circle R_1R_2 the image of the object-glass O_1O_2, the central ray cutting the axis in R the image of O formed by the eye-glass.

As the eye cannot be placed at the eye-ring, it is placed close to the eye-glass; and as the emergent rays are diverging from the axis it is useless to make the aperture of the eye-glass larger than the pupil of the eye.

192. Field of view.

If, taking the eye-glass to be of the same size as the pupil of the eye, the pupil is larger than the eye-ring as in the last figure, it will clearly be possible for objects to be seen by pencils filling the object-glass, for the breadth of the emergent parallel pencil is R_1R_2 which is less than the breadth of the pupil.

If y_o, y_e denote the radii of the object-glass and eye-glass $R_1R = \frac{f}{F} y_o$, so that in this case $y_e > \frac{f}{F} y_o$.

The entrance pupil N_1NN_2 falls behind the eye-lens, its position being given by $F_1N \,.\, F_2E = -F^2$, and its radius $N_1N = \frac{F}{f} y_e$, so

that in the case we are considering $N_1N > O_1O$; and it is clear from Fig. 121 that points within the half cone between the lines N_1O_1, N_2O_2 to the right of the point C will be seen by pencils filling the object-glass, for lines from such a point to all points of

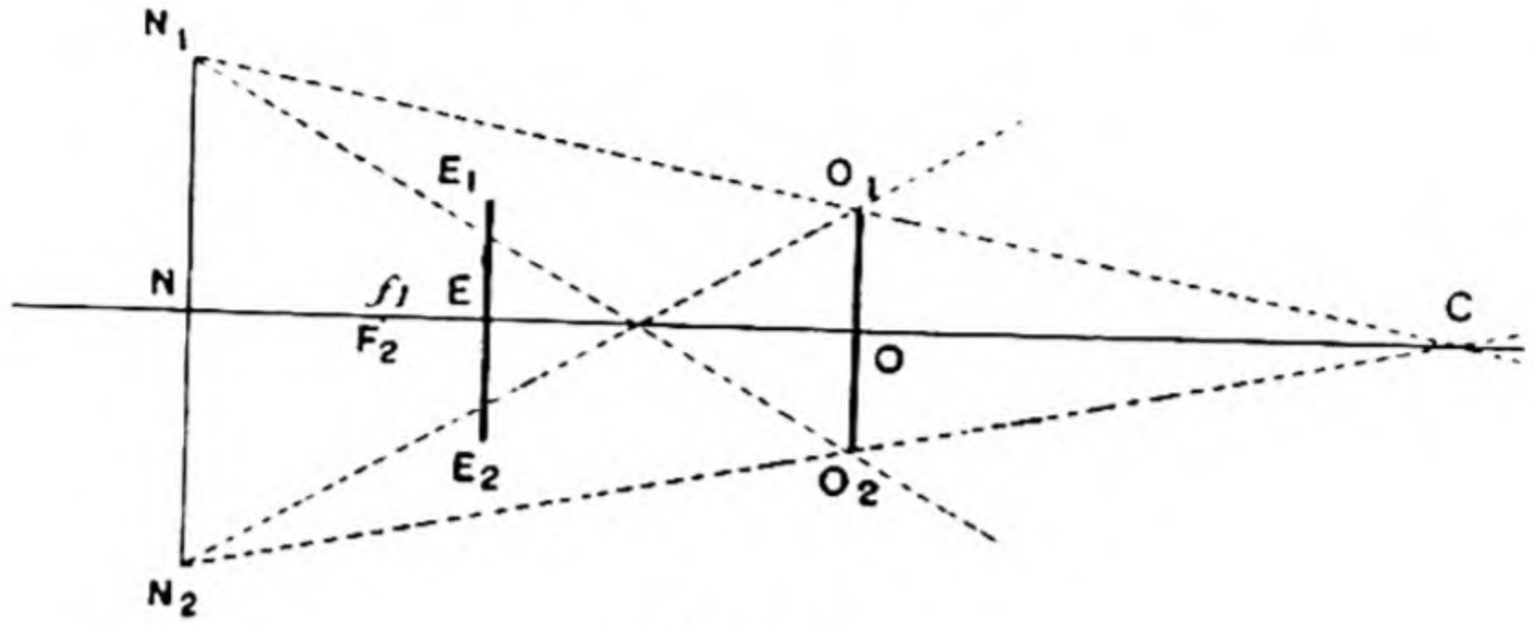

Fig. 121.

the object-glass are directed so as to fall within the entrance pupil N_1N_2, therefore they emerge through the eye-glass. The angular radius of the field of view for such pencils is the inclination of the extreme ray O_1N_1 to the axis, *i.e.*

$$\Theta = \frac{NN_1 - OO_1}{NO} = \frac{Fy_e/f - y_o}{\frac{F^2}{f} - F} = \frac{Fy_e - fy_o}{F(F-f)}.$$

The angular radius of the extreme field of view is in like manner the inclination of N_1O_2 to the axis, *i.e.*

$$\Theta' = \frac{Fy_e + fy_o}{F(F-f)}.$$

If however the eye-glass is smaller than the eye-ring, *i.e.* if $y_e < \frac{f}{F} y_o$, it will not be possible for a pencil filling the object-glass wholly to emerge from the eye-glass. In this case, since

$$N_1N = Fy_e/f,$$

the entrance pupil N_1NN_2 is clearly smaller than the object-glass O_1OO_2 and the condition that, if incident rays are to emerge through the eye-glass, they must be directed at incidence so as to fall within N_1N_2 makes it further evident that the whole of a pencil filling the object-glass cannot reach the eye. The pupil can be filled

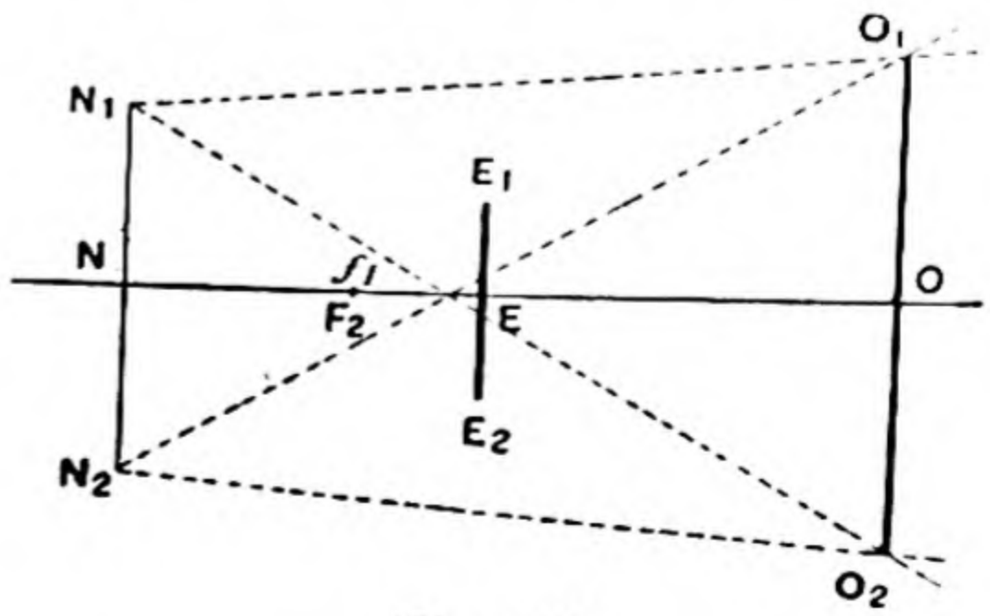

Fig. 122.

in this case by pencils that partly fill the object-glass, and the extreme inclination to the axis of such pencils at incidence is the inclination of N_1O_1, *i.e.* the angular radius of the field of view for pencils that fill the pupil is given by

$$\Theta_1 = \frac{OO_1 - NN_1}{NO} = \frac{fy_o - Fy_e}{F(F-f)}.$$

As before the inclination of N_1O_2 to the axis defines the limits of a wider field of view, and objects between the two cones will be seen by pencils only partially filling the pupil, and therefore appearing less bright.

The angular radius of the extreme field is Θ' with the same value as before. In this telescope it is impossible to place a stop to cut off the ragged edge of the field, because the common focal plane falls outside the instrument.

193. Adjustment for vision at a finite distance.

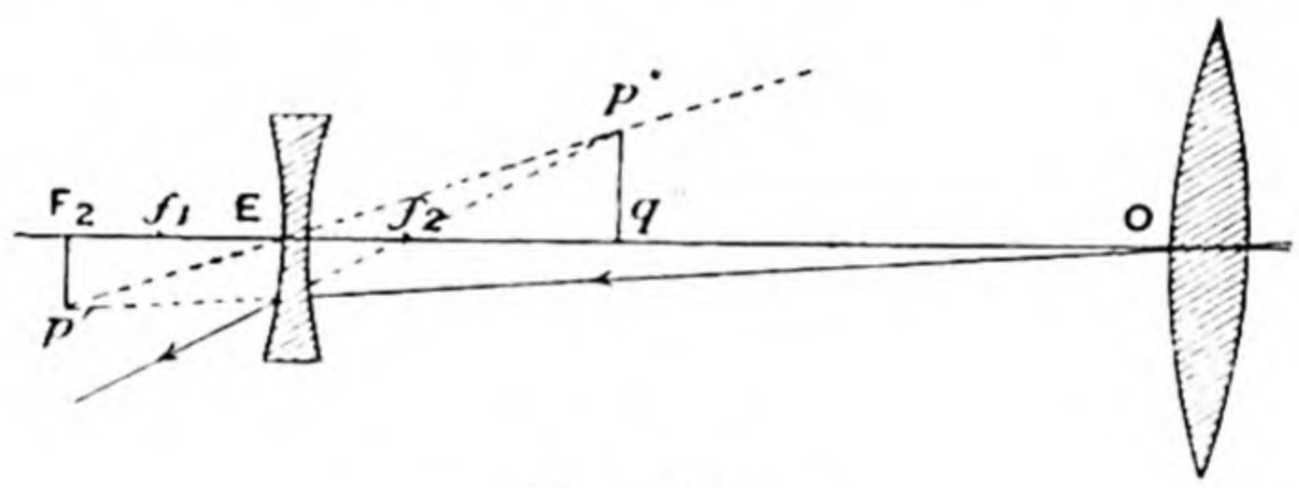

Fig. 123.

If the eye-glass be pushed in a distance x, it forms a virtual image $p''q''$ of the image $p'F_2$ formed by the object-glass, and if D be the distance of distinct vision

$$D = Eq'' = Ef_2 + f_2q'' = f + \frac{f^2}{x}.$$

The magnifying power in this case is

$$m = \frac{p''Eq''}{p'OF_2} = \frac{p''q''}{p'F_2} \cdot \frac{F_2O}{Eq''} = \frac{F}{D} \cdot \frac{f}{x} = \frac{F}{D}\left(\frac{D}{f} - 1\right)$$
$$= \frac{F}{f}\left(1 - \frac{f}{D}\right).$$

This expression shews that a short-sighted person may have to forgo an appreciable fraction of the magnifying power of the instrument. If we allow for the fact that the eye cannot get quite close to the eye-glass and take δ as its distance, the above formula becomes

$$m = \frac{F}{f}\left(1 - \frac{f+\delta}{D}\right).$$

For a very myopic eye D may be as small as 10 cms. and taking δ as 1·5 cms. and f as 3·5 cms., we see that half the magnifying power is lost.

194. Galileo's telescope is suitable for viewing terrestrial objects, and a pair of such telescopes suitably mounted form the ordinary opera or field glasses.

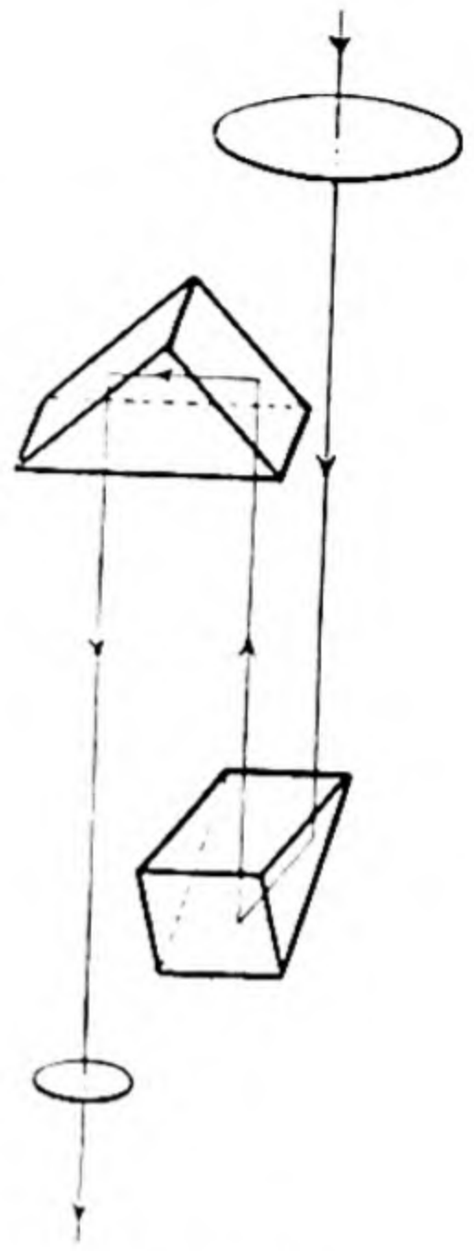

Fig. 124.

195. The Prism Binocular.

An arrangement by which a pair of prisms can be used together with the lenses of a common astronomical telescope for producing an erect instead of an inverted image was invented and patented by J. Porro in 1852*. The advantage of the arrangement is that the properties which the astronomical telescope possesses as regards field of view are obtained without the inconvenient length of instrument that arises when an erecting eye-piece is used.

It is clear from Fig. 125 that if a rectangular prism be used as a total reflection prism, the light being incident normally on the hypotenuse face, the result of reflection is to invert images in the direction at right angles to the edge of the prism. And if two such prisms are used with their edges at right angles as in Fig. 124 images will be completely inverted. A pair of instruments formed by convergent lenses as in Fig. 124 with simple or compound object-glass and eye-piece constitute the *prism binocular*, and the best field-glasses are of this form.

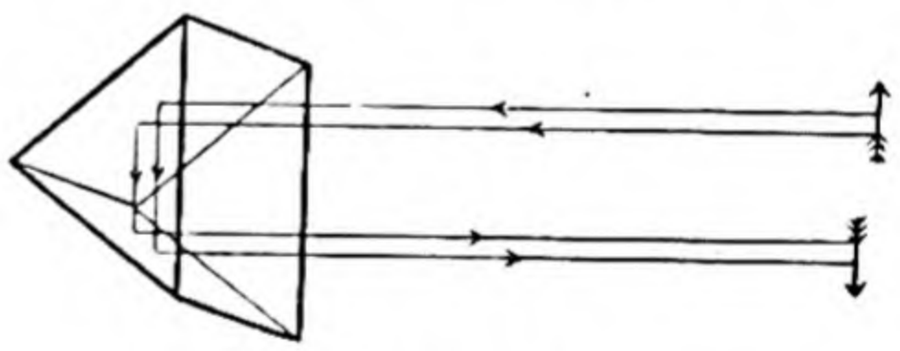

Fig. 125.

196. Examples. 1. If opera glasses are used by a short-sighted person, must the lenses originally in normal adjustment be screwed nearer together or further apart ?
(Coll. Exam. 1910.)

2. Shew that if a Galileo's telescope is adjusted first for a short-sighted person and then for a long-sighted one, the magnifying power is greater in the latter case.
(Coll. Exam. 1891.)

* The invention is described in *Cosmos*. t. II. p. 222. Paris, 1852.

3. An opera glass when in normal adjustment for a normal eye is used by a short-sighted person whose distance of distinct vision is Δ. Shew that the objects which he sees distinctly are under the angular magnification

$$\frac{F}{f} - \frac{(F-f)^2}{F\Delta},$$

where F, f are the numerical focal lengths of the object-glass and eye-glass.
(M. T. 1909.)

4. If the magnifying power of a Galileo's telescope be 6, the length 6 inches, and the radii of the object and eye glasses $\frac{1}{2}$ and $\frac{1}{1\frac{1}{2}}$ inch respectively, find those of the eye-ring and entrance pupil and shew that the angular diameter of the field of view is about 3° 11′. (Coll. Exam. 1905.)

5. A person looks directly into a mirror with an ordinary pair of opera glasses. Examine whether he can see his eyes and whether they look larger or smaller than without the glasses. (Coll. Exam. 1905.)

6. The two lenses in an opera glass are three inches apart. Print placed very near the object-glass appears two-thirds its original size when viewed by an eye at a considerable distance, and when placed close to the eye-glass and viewed through the instrument by an eye at a considerable distance the print appears one and a half times its original size. Determine the focal lengths of the eye-glass and the object-glass and state whether the glasses are focused for distant objects. (M. T. 1902.)

197. Reflecting telescopes.

In reflecting telescopes a concave mirror is used in place of an object-glass, the mirror forms an image of a distant object in its focal plane and this may be viewed by an eye-piece.

198. In **Newton's telescope** the rays converging from the concave mirror are intercepted by a small plane mirror at an inclination of 45° to the axis so that the image can be viewed through an eye-piece whose axis is at right angles to the axis of the instrument.

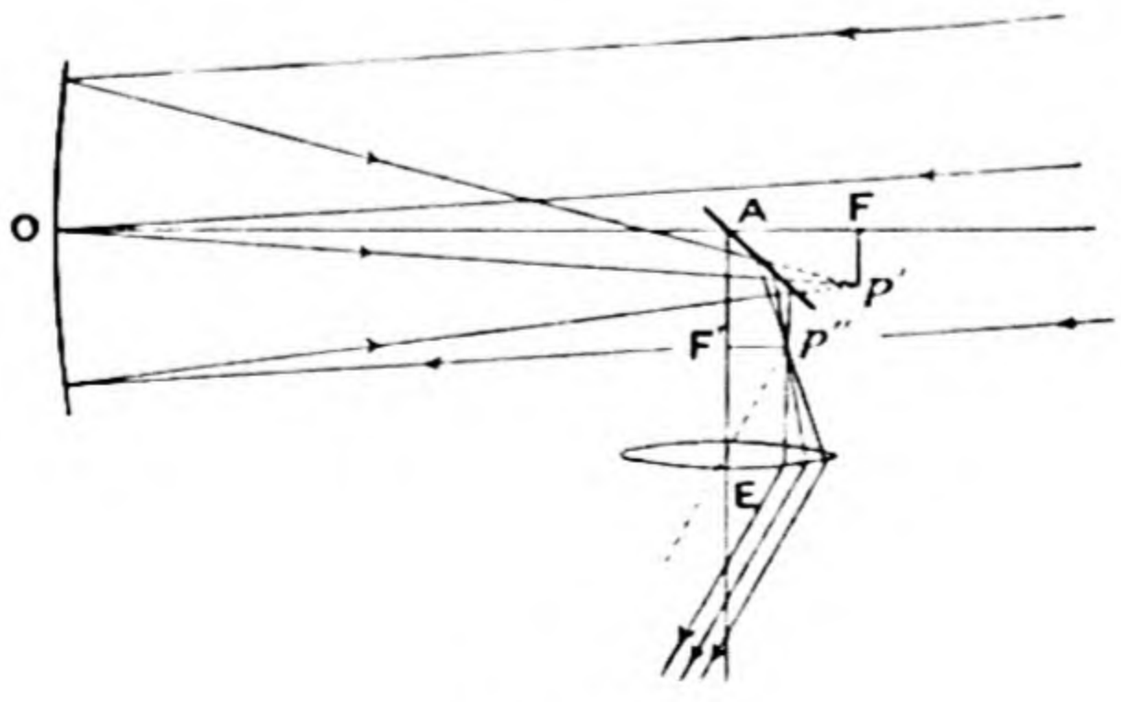

Fig. 126.

A pencil of rays making a small angle with the axis is reflected by the concave mirror towards p' on its focal plane, but strikes the small plane mirror A and converges to p'', the image of p' with regard to the small mirror; this image p'' in the plane $F'p''$ is then viewed through an eye-piece E, and in normal adjustment the first focus of the eye-piece must coincide with F' the image in the small mirror of the focus F of the large mirror. The magnifying power is the ratio of the angles $p''EF'$ and $p'OF$, taking the latter as the angle that the object would subtend at the eye if viewed directly. But $F'p'' = Fp'$, so that

$$m = \frac{FO}{F'E} = \frac{F}{f},$$

where F, f are the focal lengths of the concave mirror and the eye-piece.

199. In **Herschel's telescope** the mirror is placed obliquely to the tube and the image formed in the focal plane is formed near the edge of the tube and viewed directly through an eye-piece.

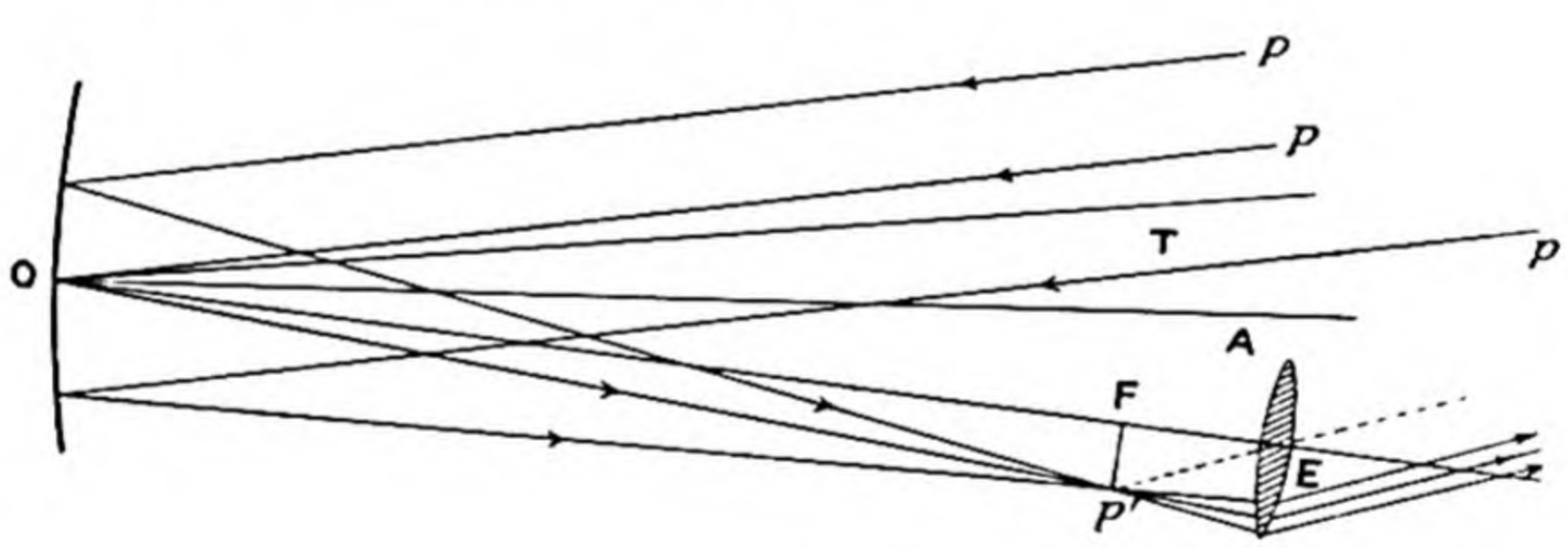

Fig. 127.

OT is the axis of the tube, OA that of the mirror, the angle between them being small; the image of a distant point p is formed at p' in the focal plane and viewed through the eye-piece E. The magnifying power is again $-F/f$, as in the astronomical telescope.

200. Cassegrain's telescope.

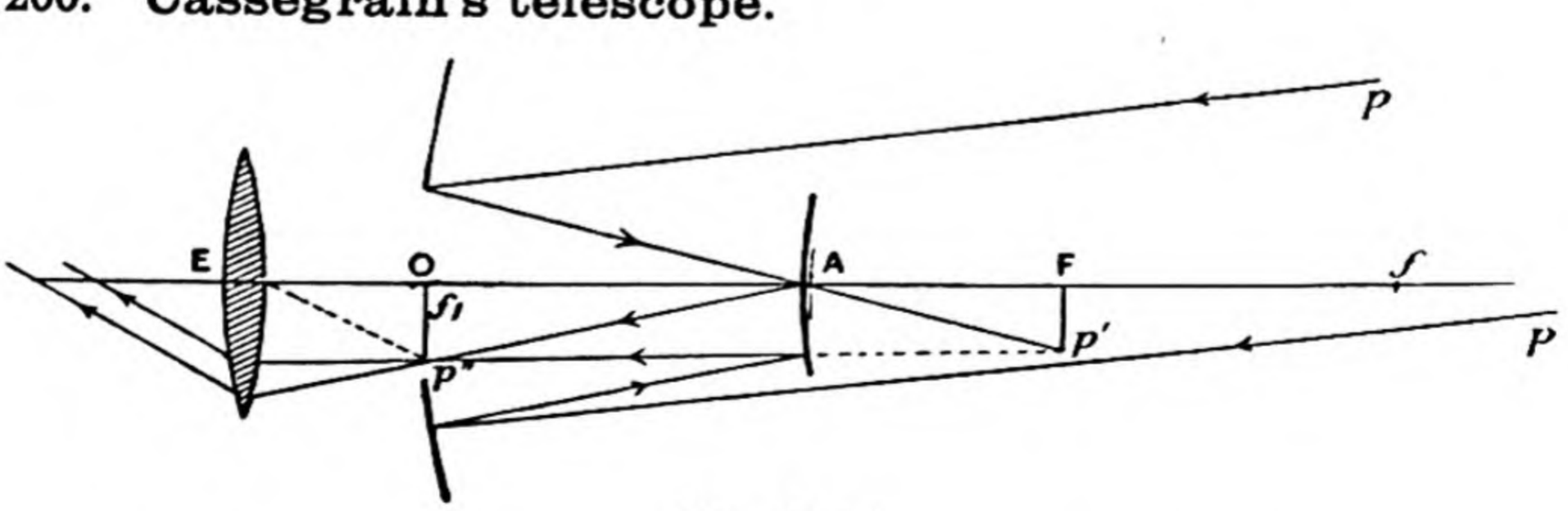

Fig. 128.

In this instrument a large concave mirror causes rays from a distant point p to converge towards a point p' in its focal plane, but the rays are intercepted

by a smaller mirror A convex to the light and brought to a focus at p'' the image of p' in the convex mirror. There is an aperture at the vertex of the large mirror through which the image p'' is viewed by a single lens or an eye-piece E. In normal adjustment the first focal plane of the eye-piece contains the image p''.

Since the angle the incident rays make with the axis is the angle which $p'F$ subtends at the vertex O of the mirror, therefore the magnifying power

$$= \frac{p''Ef_1}{p'OF}$$

$$= \frac{p''f_1}{p'F} \cdot \frac{FO}{f_1E} = \frac{f_m F}{\alpha f_e},$$

where F, f_m, f_e are the focal lengths of the large and small mirrors and the eye-piece and α is the distance between the foci F, f of the two mirrors.

201. **Gregory's telescope** is similar to Cassegrain's, but the small mirror is concave instead of convex to the light.

202. The **Compound Microscope** in its simplest form consists of a convergent object-glass called the *objective* which forms a real magnified image of a small object a little beyond its first focus, the image being formed on the other side of its second focus—and a convergent eye-glass called the *ocular* so placed as to form a magnified virtual image of the former image.

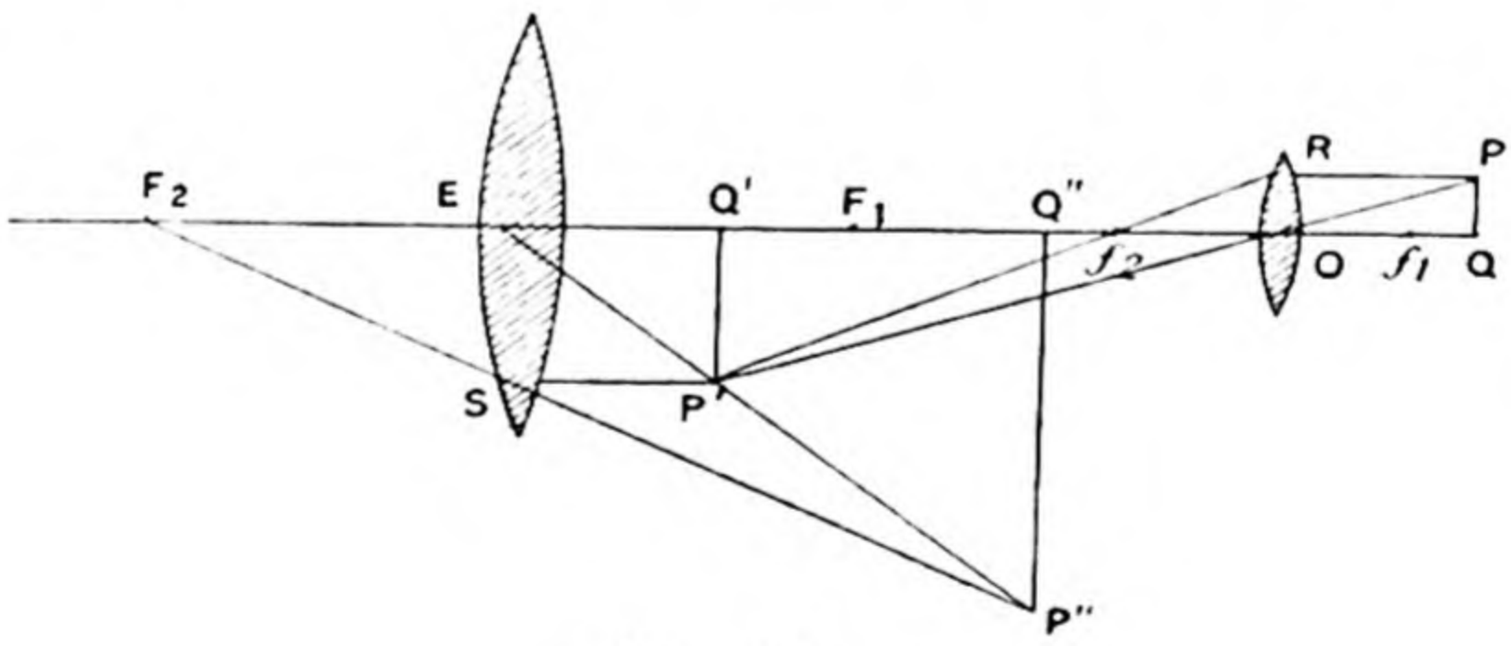

Fig. 129.

f_1, f_2 are the foci of the object-glass O and F_1, F_2 those of the eye-glass E. PQ is the object and the first image $P'Q'$ is determined by taking rays PR parallel to the axis and PO through the centre of the lens. The second image $P''Q''$ is determined by taking rays $P'S$ parallel to the axis and $P'E$ through the centre of the eye-glass. If f, F are the numerical focal lengths of the object-glass and eye-glass and a the distance between them, the power of the instrument is

$$-\frac{1}{f} - \frac{1}{F} + \frac{a}{fF}.$$

The linear magnification may be expressed in terms of the distance D of the final image from the ocular; thus, we have

$$m = \frac{P''Q''}{P'Q'} \cdot \frac{P'Q'}{PQ} = \frac{D}{EQ'} \cdot \frac{f_2 Q'}{f},$$

where $$\frac{1}{D} - \frac{1}{EQ'} = -\frac{1}{F}, \text{ so that } EQ' = \frac{FD}{F+D},$$

and

$$m = -\frac{F+D}{Ff}\left(a - f - \frac{FD}{F+D}\right)$$

$$= -\left(\frac{a}{f} - 1\right) - D\left(\frac{a}{fF} - \frac{1}{F} - \frac{1}{f}\right).$$

The magnification is therefore increased, by increasing the distance between the lenses and decreasing the focal length.

203. In practice microscopes contain many lenses adjusted so as to reduce spherical and chromatic aberration. The first lens of the objective is always of very short focal length and rays are incident on it making widely divergent angles with the axis. It has generally one plane face and is followed by other compound lenses of longer focal length each so arranged as to correct some of the defects of the previous lenses. In objectives of high power the lens nearest the object is sometimes a hemisphere with its plane face towards the object with a film of cedar-wood oil between the lens and the glass slide in which the object is mounted. The refractive index of cedar-wood oil is approximately the same as that of glass. The advantage of this method of *immersion* as it is called is that the divergence of the rays emerging from the glass-slide into air is eliminated, so that the object-glass receives more rays from the object than it would if oil were not used. For if the oil were not used some of the rays which strike the object-glass when the oil is present, emerging from the slide into air, would be bent outwards so far as to miss the object-glass.

204. Forms of object-glasses.

The construction of an object-glass depends on the particular purpose for which it is to be used.

In a photographic camera the object-glass produces a real image of an extended object on a sensitive plate. The violet end of the spectrum is the active

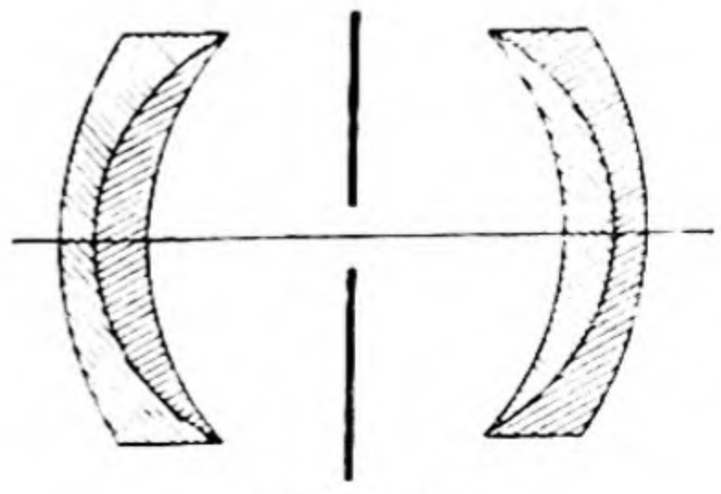

Fig. 130

one, and hence the object-glass must be achromatized so that the visual image coincides with that formed by the active rays.

Flatness of image and absence of distortion must be secured so far as possible. This is attained by combining two similar compound lenses and inserting a stop between them. Such a combination is said to be *orthoscopic* or *rectilinear*, it is free from distortion as each half of the combination produces an equal and opposite amount of distortion. A wide angle for the field of view is required and as wide an aperture as possible in order to admit sufficient light. These latter requirements cannot both be obtained at once, for a wide angle means oblique incidence and considerable aberration unless the rays are restricted by a stop to pass through a small aperture on the axis.

205. In *telephotography* what is required is to produce on the sensitive plate as large an image as possible of a distant object. This means that the focal length of the object-glass must be as great as possible, and this would necessitate an inconvenient length of camera. This is obviated by the use of a *telephotographic objective*, which consists of a combination of a convergent and a divergent lens with a space between them.

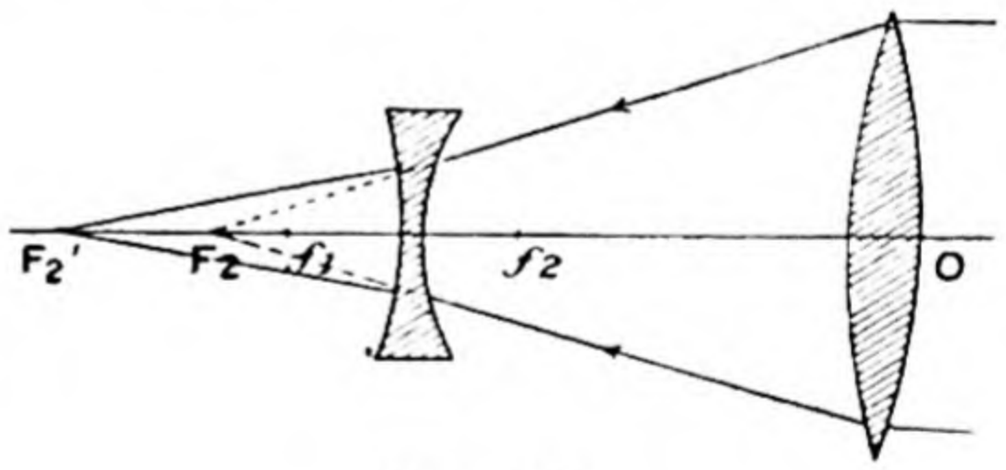

Fig. 131.

Thus F_2 is the second focus of the convergent lens behind the first focus f_1 of the divergent lens, and F_2' is the second focus of the combination in or near the position of the sensitive plate. Without the divergent lens the length of the camera would be approximately F_2O; the divergent lens increases it slightly to $F_2'O$ but gives a greatly increased focal length to the combination, for the power is decreased to

$$\frac{1}{F} - \frac{1}{f} - \frac{a}{Ff}$$

if F, f are the focal lengths and a the distance between the lenses. In fact the increased focal length is obtained mainly by the shifting of the second unit point from somewhere close to O to a distance in front of the convergent lens.

EXAMPLES.

1. Draw a pencil of rays passing through Newton's telescope, fitted with Ramsden's eye-piece, supposing the rays to emerge parallel.

(Coll. Exam. 1902.)

2. A telescope consists of a convex object-glass of focal length F, and an eye-piece composed of two convex lenses of focal lengths f_1 and f_2 separated by a fixed distance a. Find the whole length of the telescope when adjusted for a distant object to be seen by an eye which sees clearly at distance d. Shew that when d is very large the length is

$$a+F+\frac{f_1 f_2-af_1}{f_1+f_2-a}.$$

In the latter case find the angular magnifying power. (Coll. Exam. 1913.)

3. If m denote the ratio of the focal lengths and n that of the radii of the apertures of the object-glass and eye-glass of an astronomical telescope, then if $m<n$ no distant objects can be viewed by pencils filling the object-glass, but it may be possible so to view small objects on the axis whose distance from the object-glass does not exceed $\frac{lmn}{n-m}$ where l is the distance between the lenses. Discuss the existence of a lower limit to the distance of the object for which such vision is possible. (M. T. 1903.)

4. Prove that, if a common astronomical telescope is adjusted to suit a person who sees best at distance Δ, then the magnifying power will be diminished numerically by f/Δ, and the angular radius of the field of view will be increased by $(y_e+y_o)f^2/\Delta(f+F)^2$, provided that higher powers of $1/\Delta$ are neglected; where y_e, y_o are the radii and f, F the focal lengths of the eye-glass and object-glass respectively and the eye is placed at the eye-ring.

(M. T. 1904.)

5. A telescope consists of a thin object-glass of focal length F and a movable eye-piece of focal length f; the eye is always placed at the image of the object-glass formed by the rest of the telescope. Shew that, if M_1 be the magnifying power when adjusted for a distance of distinct vision Δ_1, and M_2, Δ_2 be corresponding quantities, then

$$(M_1\Delta_1-M_2\Delta_2)F=(M_2^2\Delta_2-M_1^2\Delta_1)f=M_1M_2(M_1-M_2)\Delta_1\Delta_2.$$

(M. T. 1896.)

6. A simple astronomical telescope is adjusted so as to view an object distant u from the object-glass, when d is the distance of distinct vision and the eye is placed close to the eye-lens. Shew that the greatest angle the object can subtend at the centre of the object-glass, so that every point of it can be seen by a complete pencil, is

$$2(1/F-1/u)\{a(1/f+1/d)-b(1/F-1/u)\}/(1/f+1/F+1/d-1/u),$$

where f, F and a, b are the focal lengths and radii of the eye-lens and object-glass respectively. (M. T. 1906.)

7. Prove that a cylinder of glass, whose length is μ times the distance between the centres of curvature of its spherical ends, acts optically as a telescope focused for distant vision. Give diagrams to represent its effect.
(St John's Coll. 1903.)

8. Prove that if f is the focal length of the Ramsden's eye-piece, and f' is the focal length when the space between the lenses is filled with a medium of refractive index μ', then

$$\frac{f'}{f} = \frac{2\mu'(\mu-1)^2}{(\mu-\mu')\{(\mu-1)(3\mu'-1)+(\mu'-1)\}},$$

where μ is the refractive index of the glass of either lens, and the lenses are plano-convex with the plane faces outwards. (Trinity Coll. 1904.)

9. The aperture of the object-glass of an astronomical telescope, of magnifying power $-m$, is $2c$. The single-lens eye-piece is removed, and replaced by a Ramsden's eye-piece of equal power, the apertures of whose eye-glass and field-glass are $2a$ and $2b$ respectively. Prove that the radius of the circular stop suitable for the new arrangement is the lesser of the quantities

$$(3mb-c)/(3m+1), \text{ and } 3(ma-c)/(m+3). \quad \text{(M. T. 1901.)}$$

10. A Ramsden's eye-piece is fitted in an astronomical telescope with achromatic object-glass and brought to normal adjustment. A ray from the centre of the object-glass is refracted through the eye-piece. Prove that if Δ is the dispersion of this ray for two colours, and if Δ' is the dispersion for those two colours when the eye-piece is replaced by a single lens such that the magnifying power m of the telescope is unaltered,

$$\frac{\Delta}{\Delta'} = \frac{3m+5}{6m+6}. \quad \text{(Coll. Exam. 1910.)}$$

11. A telescope consists of a convex object-glass of focal length F, and an eye-piece of any number of lenses of which the equivalent focal length is f, and is in normal adjustment. Prove that (i) the linear magnification of all small objects, which are perpendicular to the axis, is the same, and is equal to f/F; (ii) the distances of any object from the first principal focus of the object-glass and of its image from the second principal focus of the eye-piece are always in the ratio $F^2 : f^2$. (Trinity Coll. 1890.)

12. Shew that, if the eye be at a distance x from the first principal focus of any refracting telescope and the object be at a distance y from the second principal focus, the magnifying power is $fy/(xy-f^2)$ whatever be the distance of distinct vision, f being the focal length of the telescope, where x, y are measured positively away from the telescope, and y is great compared with the length of the telescope. (Coll. Exam. 1899.)

13. In a microscope formed of lenses whose relative positions are fixed, shew that, if the instrument be pushed nearer to the object by a small distance δ, the change in the position of the image is $m^2\delta$, where m is the magnification and the change in the magnification is $m^2\delta/F$, where F is the focal length of the instrument. (Coll. Exam. 1895.)

14. A microscope is made of two simple lenses, focal lengths f, F, at a fixed distance apart equal to $f+F$. If it is in adjustment for a person whose distance of distinct vision is d, prove that the object must be moved through a distance $(d'-d)\,\frac{f^2}{F^2}$ to suit a person whose distance of distinct vision is d', where $d'-d$ is a small quantity. (Coll. Exam. 1895.)

15. Explain how a camera lens consisting of a convex lens with a concave lens behind it, whose distance is adjustable, can produce a telephotographic effect, *i.e.* produce a large image of a distant object without excessive length of camera. (M. T. 1911.)

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